A New Practical Framework for the Stability Analysis of Perturbed Saddle-point Problems and Applications

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Formulation of perturbed saddle point problems (PSPPs)

Notation

Consider the following setting:

- V and Q denote Hilbert spaces,
- for the norms $\|\cdot\|_V$ and $\|\cdot\|_Q$,
- induced by the scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$,

respectively.

$Y := V \times Q$ denotes their product space, endowed with the norm $\|\cdot\|_Y$:

$$\|y\|_Y^2 = (y,y)_Y = (v,v)_V + (q,q)_Q = \|v\|_V^2 + \|q\|_Q^2 \quad \forall y = (v;q) \in Y.$$

Next, consider an abstract bilinear form $\mathcal{A}((\cdot\,;\cdot),(\cdot\,;\cdot))$ on $Y \times Y$:

$$\mathcal{A}((u;p),(v;q)) := a(u,v) + b(v,p) + b(u,q) - c(p,q) \qquad (1)$$

composed from the bilinear forms:

- $a(\cdot,\cdot)$ on $V \times V$ and $c(\cdot,\cdot)$ on $Q \times Q$, being SPSD and bounded,
- $b(\cdot,\cdot)$ on $V \times Q$, being bounded.

Abstract perturbed saddle-point problem (PSPP)

Each of these bilinear forms defines a linear operator as follows:

$$A : V \to V' : \langle Au, v \rangle_{V' \times V} = a(u, v), \qquad \forall u, v \in V, \tag{2a}$$

$$C : Q \to Q' : \langle Cp, q \rangle_{Q' \times Q} = c(p, q), \qquad \forall p, q \in Q, \tag{2b}$$

$$B : V \to Q' : \langle Bv, q \rangle_{Q' \times Q} = b(v, q), \qquad \forall v \in V, \forall q \in Q. \tag{2c}$$

For the bilinear form in (1), consider the perturbed saddle-point problem

$$\mathcal{A}((u; p), (v; q)) = \mathcal{F}((v; q)) \qquad \forall v \in V, \forall q \in Q, \tag{3}$$

which for $x = (u; p) \in Y$ we write as $\mathcal{A}(x, y) = \mathcal{F}(y)$, $\forall y = (v; q) \in Y$, or, in operator form, as

$$\mathcal{A}x = \begin{pmatrix} A & B^* \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \mathcal{F}. \tag{4}$$

$$\mathcal{A} : Y \to Y' : \langle \mathcal{A}x, y \rangle_{Y' \times Y} = \mathcal{A}(x, y), \qquad \forall x, y \in Y, \tag{5a}$$

$$\mathcal{F} \in Y' : \mathcal{F}(y) = \langle \mathcal{F}, y \rangle_{Y' \times Y} \qquad \forall y \in Y. \tag{5b}$$

Babuska's and Brezzi's conditions for stability of PSPPs

Babuska's theorem

The abstract variational problem (3) is well-posed under the necessary and sufficient conditions (6) and (7) given in the following theorem.

Theorem 1 [Babuška, 1971]

Let $\mathcal{F} \in Y'$ be a bounded linear functional. Then the saddle-point problem (3) is well-posed if and only if there exist positive constants $\bar{C}$ and α for which the conditions

$$\mathcal{A}(x, y) \leq \bar{C}\|x\|_Y \|y\|_Y \qquad \forall x, y \in Y, \tag{6}$$

$$\inf_{x \in Y} \sup_{y \in Y} \frac{\mathcal{A}(x, y)}{\|x\|_Y \|y\|_Y} \geq \underline{\alpha} > 0 \tag{7}$$

hold. The solution x then satisfies the stability estimate

$$\|x\|_Y \leq \frac{1}{\underline{\alpha}} \sup_{y \in Y} \frac{\mathcal{F}(y)}{\|y\|_Y} =: \frac{1}{\underline{\alpha}} \|\mathcal{F}\|_{Y'}.$$

Brezzi's theorem

For the classical saddle-point problem, i.e., $c(\cdot,\cdot) \equiv 0$, we have the following theorem which we formulate here under the condition

$$\mathrm{Ker}(B^T) := \{q \in Q : b(v,q) = 0 \ \forall v \in V\} = \{0\}. \tag{8}$$

Theorem 2 [Brezzi, 1974], [Boffi, Brezzi, Fortin, 2013]

Assume that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous on $V \times V$ and on $V \times Q$, respectively, $a(\cdot,\cdot)$ is symmetric positive semidefinite, and also that

$$a(v,v) \geq \underline{C}_a \|v\|_V, \qquad \forall v \in \mathrm{Ker}(B), \tag{9}$$

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta > 0, \tag{10}$$

hold. Then the classical saddle-point problem (Problem (3) with $c(\cdot,\cdot) \equiv 0$) is well-posed.

What can we say if $c(\cdot,\cdot) \not\equiv 0$?

If $c(\cdot,\cdot) \not\equiv 0$ an additional assumption can be used to ensure the well-posedness of the PSPP. Consider the following auxiliary problem:

$$\epsilon(p_0, q)_Q + c(p_0, q) = -c(p^{\perp}, q), \qquad \forall q \in \mathrm{Ker}(B^T) \tag{11}$$

Assumption 1 [Brezzi & Fortin, 1991]

There exists a $\gamma_0 > 0$ such that for every $p^{\perp} \in (\mathrm{Ker}(B^T))^{\perp}$ and every $\epsilon > 0$ the solution $p_0 \in \mathrm{Ker}(B^T)$ of (11) satisfies $\gamma_0 \|p_0\|_Q \leq \|p^{\perp}\|_Q$.

Theorem 3 [Brezzi & Fortin, 1991]

Let the conditions of Theorem 2 be satisfied and let $c(\cdot,\cdot)$ be continuous and SPSD. Then, under Assumption 1, Problem (3) for every $f \in V'$ and every $g \in \mathrm{Im}(B)$ and $\mathcal{F}(y) := \langle f, v\rangle_{V'\times V} + \langle g, q\rangle_{Q'\times Q}$ has a unique solution $x = (u; p)$ in $Y = V \times Q/M$ where $M = \mathrm{Ker}(B^T) \cap \mathrm{Ker}(C)$. Moreover, for a constant $C(\bar{C}_a, \bar{C}_c, \underline{C}_a, \beta, \gamma_0)$ there holds the estimate

$$\|u\|_V + \|p\|_{Q/\mathrm{Ker}(B^T)} \leq C(\|f\|_{V'} + \|g\|_{Q'}).$$

Can we dispense with Assumption 1?

In order to ensure the boundedness (continuity) of $c(\cdot,\cdot)$, it is natural to include the contribution of $c(\cdot,\cdot)$ in the norm $\|\cdot\|_Q$, e.g., by defining

$$\|q\|_Q^2 = |q|_Q^2 + t^2 c(q,q), \qquad \forall q \in Q, \tag{12}$$

for a proper seminorm or norm $|\cdot|_Q$ and a parameter $t \in [0,1]$.

Then the stability of the PSPP can be proven under Brezzi's conditions for the classical saddle-point problem and the additional condition

$$\inf_{u\in V} \sup_{(v;q)\in V\times Q} \frac{a(u,v)+b(u,q)}{|||(v;q)|||} \geq \gamma > 0, \tag{13}$$

where $|||(v;q)|||^2 := \|v\|_V^2 + |q|_Q^2 + t^2 c(q,q), \quad t \in [0,1]$.

Theorem 4 [Braess, 1996]

Assume that conditions (9) and (10) are fulfilled and (13) holds with $\gamma > 0$ for some $t > 0$. Then the PSPP (3) is stable under the norm $\|\cdot\|_Y := |||\cdot|||$ and the constant $\underline{\alpha}$ in (7) depends only on β, $\underline{C}_a$, γ, t.

A new framework for the stability analysis of PSPPs

Our goal

- Develop Brezzi like condition for PSPPs which only need to check two conditions: the coercivity of $a(\cdot,\cdot)$ and the small inf-sup condition for $b(\cdot,\cdot)$.
- The stability constants are uniform with respect to the parameters appeared in the bilinear forms $a(\cdot,\cdot), b(\cdot,\cdot)$ and $c(\cdot,\cdot)$

Key idea: Norm splittings

For fixed $t \in (0, 1]$, the norm (12) is equivalent to

$$\|q\|_Q^2 := |q|_Q^2 + c(q, q) =: \langle \bar{Q} q, q \rangle_{Q' \times Q} \tag{14}$$

where $\bar{Q} : Q \to Q'$ is a linear operator.

Now we introduce the following splitting of the norm $\| \cdot \|_V$ defined by

$$\|v\|_V^2 := |v|_V^2 + |v|_b^2 \tag{15}$$

where $| \cdot |_V$ is a proper seminorm, which is a norm on $\mathrm{Ker}(B)$ satisfying

$$|v|_V^2 \eqsim a(v, v), \qquad \forall v \in \mathrm{Ker}(B)$$

and $| \cdot |_b$ is defined by

$$|v|_b^2 := \langle Bv, \bar{Q}^{-1} Bv \rangle_{Q' \times Q} = \|Bv\|_{Q'}^2. \tag{16}$$

Then $\bar{Q}^{-1} : Q' \to Q$ is an isometric isomorphism (Riesz isomorphism),

$$\|\bar{Q}^{-1} Bv\|_Q^2 = \|Bv\|_{Q'}^2 = \langle \bar{Q} \bar{Q}^{-1} Bv, \bar{Q}^{-1} Bv \rangle_{Q' \times Q} = \langle Bv, \bar{Q}^{-1} Bv \rangle_{Q' \times Q}.$$

Key idea: Fitted norms

Remark

Note that both $|\cdot|_V$ *and* $|\cdot|_b$ *can be seminorms as long as they add up to a full norm. Likewise, only the sum of the seminorms* $|\cdot|_Q$ *and* $c(\cdot,\cdot)$ *has to define a norm.*

In order to present our main theoretical result, we make the definition.

Definition 1

Two norms $\|\cdot\|_Q$ and $\|\cdot\|_V$ on the Hilbert spaces Q and V are called fitted if they satisfy the splittings (14) and (15), respectively, where $|\cdot|_Q$ is a seminorm on Q and $|\cdot|_V$ and $|\cdot|_b$ are seminorms on V, the latter defined according to (16).

Main stability result

Theorem 5 [H., Kraus, Lymbery, Philo, 2021]

Let $\|\cdot\|_V$ and $\|\cdot\|_Q$ be fitted norms according to Definition 1, which immediately implies the continuity of $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ in these norms with $\bar{C}_b = 1$ and $\bar{C}_c = 1$, cf. (14)–(16). Consider the bilinear form $\mathcal{A}((\cdot;\cdot),(\cdot;\cdot))$ defined in (1) where $a(\cdot,\cdot)$ is SPSD and continuous, and $c(\cdot,\cdot)$ is SPSD. Assume that $a(\cdot,\cdot)$ satisfies the coercivity estimate

$$a(v,v) \geq \underline{C}_a |v|_V^2, \qquad \forall v \in V, \tag{17}$$

and $b(\cdot,\cdot)$ the inf-sup-type condition that there exists a constant $\underline{\beta} > 0$ s.t.

$$\sup_{\substack{v \in V \\ v \neq 0}} \frac{b(v,q)}{\|v\|_V} \geq \underline{\beta} |q|_Q, \qquad \forall q \in Q. \tag{18}$$

Then, $\mathcal{A}((\cdot;\cdot),(\cdot;\cdot))$ is continuous and inf-sup stable under the combined norm $\|\cdot\|_Y$ defined by $\|y\|_Y^2 = \|v\|_V^2 + \|q\|_Q^2$, $\forall y = (v;q) \in Y = V \times Q$.

Various applications

More details and various applications of the presented framework including

- generalized Poisson and generalized Stokes equations,
- Stokes-Darcy interface problem,
- vector Laplace equation (Maxwell),
- various formulations of Biot's model

can be found in

Q. Hong, J. Kraus, M. Lymbery, F. Philo: A new practical framework for the stability analysis of perturbed saddle-point problems and applications. Mathematics of Computation, 2023, Vol. 92, 607-634.

Application to operator preconditioning in vector Laplacian equation

Vector Laplace equation example: preliminaries

Mixed variational formulation of the vector Laplace equation: find $\boldsymbol{p} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega), \boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div}, \Omega)$, such that

$$
\begin{aligned}
(\alpha \boldsymbol{p}, \boldsymbol{q}) - (\boldsymbol{u}, \mathrm{curl}\boldsymbol{q}) &= 0, \qquad \forall \boldsymbol{q} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega),\\
-(\mathrm{curl}\boldsymbol{p}, \boldsymbol{v}) - (\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{v}) &= (f, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \Omega),
\end{aligned}
$$

where α is a positive scalar.

Vector Laplace equation example: preliminaries

Mixed variational formulation of the vector Laplace equation: find $\boldsymbol{p} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega), \boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div}, \Omega)$, such that

$$\begin{aligned} (\alpha \boldsymbol{p}, \boldsymbol{q}) - (\boldsymbol{u}, \mathrm{curl}\boldsymbol{q}) &= 0, \qquad \forall \boldsymbol{q} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega), \\ -(\mathrm{curl}\boldsymbol{p}, \boldsymbol{v}) - (\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{v}) &= (f, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \Omega), \end{aligned}$$

where α is a positive scalar.

We rewrite the above equations as

$$\begin{aligned} (\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{v}) + (\mathrm{curl}\boldsymbol{p}, \boldsymbol{v}) &= -(f, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \Omega), \\ (\boldsymbol{u}, \mathrm{curl}\boldsymbol{q}) - (\alpha \boldsymbol{p}, \boldsymbol{q}) &= 0, \qquad \forall \boldsymbol{q} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega). \end{aligned}$$

Vector Laplace equation: bilinear forms

The bilinear forms that define $\mathcal{A}((\cdot;\cdot),(\cdot;\cdot))$ are

$$a(\boldsymbol{u},\boldsymbol{v}) := (\mathrm{div}\boldsymbol{u},\mathrm{div}\boldsymbol{v}), \qquad \forall \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V},$$
$$b(\boldsymbol{v},\boldsymbol{p}) := (\mathrm{curl}\boldsymbol{p},\boldsymbol{v}), \qquad \forall \boldsymbol{v}\in\boldsymbol{V},\forall\boldsymbol{p}\in\boldsymbol{Q},$$
$$c(\boldsymbol{p},\boldsymbol{q}) := (\alpha\boldsymbol{p},\boldsymbol{q}), \qquad \forall \boldsymbol{p},\boldsymbol{q}\in\boldsymbol{Q},$$

where $\boldsymbol{V}=\boldsymbol{H}_0(\mathrm{div},\Omega)$, $\boldsymbol{Q}=\boldsymbol{H}_0(\mathrm{curl},\Omega)$.

We fix $|\cdot|_Q$ and $|\cdot|_V$ to be

$$|\boldsymbol{q}|_Q^2 := ((\alpha+1)\mathrm{curl}\boldsymbol{q},\mathrm{curl}\boldsymbol{q}), \qquad \forall \boldsymbol{q}\in\boldsymbol{Q},$$
$$|\boldsymbol{v}|_V^2 := (\mathrm{div}\boldsymbol{v},\mathrm{div}\boldsymbol{v}), \qquad \forall \boldsymbol{v}\in\boldsymbol{V}.$$

As before, $a(\boldsymbol{v},\boldsymbol{v})\geq|\boldsymbol{v}|_V^2$ for all $\boldsymbol{v}\in\boldsymbol{V}$, that is, (17) is satisfied with $\underline{C}_a=1$.

Vector Laplace equation: small inf-sup condition

In addition, noting that $B := \mathrm{curl}^* : \boldsymbol{V} \to \boldsymbol{Q}'$, we have

$$
\begin{aligned}
\|\boldsymbol{q}\|_Q^2 &:= |\boldsymbol{q}|_Q^2 + c(\boldsymbol{q}, \boldsymbol{q}) = ((\alpha+1)\mathrm{curl}\boldsymbol{q}, \mathrm{curl}\boldsymbol{q}) + (\alpha\boldsymbol{q}, \boldsymbol{q}) \\
&= \langle \bar{Q}\boldsymbol{q}, \boldsymbol{q}\rangle_{Q'\times Q}, \ \forall \boldsymbol{q} \in \boldsymbol{Q}, \\
\|\boldsymbol{v}\|_V^2 &:= |\boldsymbol{v}|_V^2 + \langle B\boldsymbol{v}, \bar{Q}^{-1}B\boldsymbol{v}\rangle_{Q'\times Q} \\
&= (\mathrm{div}\boldsymbol{v}, \mathrm{div}\boldsymbol{v}) + \left((\alpha I + \mathrm{curl}^*(\alpha+1)\mathrm{curl})^{-1}\mathrm{curl}^*\boldsymbol{v}, \mathrm{curl}^*\boldsymbol{v}\right), \ \forall \boldsymbol{v} \in \boldsymbol{V}.
\end{aligned}
$$

Next, for any $\boldsymbol{q} \in \boldsymbol{Q}$, choose $\boldsymbol{v}_0 = \mathrm{curl}\boldsymbol{q} \in \boldsymbol{V}$ to obtain

$$
\begin{aligned}
\|\boldsymbol{v}_0\|_V^2 &= (\mathrm{div}\,\mathrm{curl}\boldsymbol{q}, \mathrm{div}\,\mathrm{curl}\boldsymbol{q}) \\
&\quad + \left((\alpha I + \mathrm{curl}^*(\alpha+1)\mathrm{curl})^{-1}\mathrm{curl}^*\mathrm{curl}\boldsymbol{q}, \mathrm{curl}^*\mathrm{curl}\boldsymbol{q}\right) \\
&= \left((\alpha I + \mathrm{curl}^*(\alpha+1)\mathrm{curl})^{-1}\mathrm{curl}^*\mathrm{curl}\boldsymbol{q}, \mathrm{curl}^*\mathrm{curl}\boldsymbol{q}\right) \\
&\leq (\boldsymbol{q}, (\alpha+1)^{-1}\mathrm{curl}^*\mathrm{curl}\boldsymbol{q}) \\
&= ((\alpha+1)^{-1}\mathrm{curl}\boldsymbol{q}, \mathrm{curl}\boldsymbol{q})
\end{aligned}
$$

and

$$
\sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(\boldsymbol{v},\boldsymbol{q})}{\|\boldsymbol{v}\|_V} \geq \frac{b(\boldsymbol{v}_0,\boldsymbol{q})}{\|\boldsymbol{v}_0\|_V} = \frac{(\mathrm{curl}\boldsymbol{q},\mathrm{curl}\boldsymbol{q})}{\|\boldsymbol{v}_0\|_V} \geq \frac{(\mathrm{curl}\boldsymbol{q},\mathrm{curl}\boldsymbol{q})}{((\alpha+1)^{-1}\mathrm{curl}\boldsymbol{q},\mathrm{curl}\boldsymbol{q})^{\frac{1}{2}}} = |\boldsymbol{q}|_Q.
$$

Vector Laplace equation: operator preconditioner

Note that

$$\|\boldsymbol{v}\|_V^2 := (\mathrm{div}\,\boldsymbol{v}, \mathrm{div}\,\boldsymbol{v}) + \big((\alpha \boldsymbol{I} + \mathrm{curl}^*(\alpha+1)\mathrm{curl})^{-1}\mathrm{curl}^*\boldsymbol{v}, \mathrm{curl}^*\boldsymbol{v}\big)$$

is equivalent to

$$(\mathrm{div}\,\boldsymbol{v}, \mathrm{div}\,\boldsymbol{v}) + ((\alpha+1)^{-1}\boldsymbol{v}, \boldsymbol{v}).$$

Hence we obtain the following norm-equivalent operator preconditioner:

$$\mathcal{B} := \begin{bmatrix} ((\alpha+1)^{-1}\boldsymbol{I} - \nabla\mathrm{div})^{-1} & \\ & (\alpha\boldsymbol{I} + (\alpha+1)\mathrm{curl}^*\mathrm{curl})^{-1} \end{bmatrix}.$$

Applications to operator preconditioning in poromechanics

Examples in poromechanics: preliminaries

In the following, we will make use of the following classical inf-sup conditions, see, e.g., [Brezzi & Fortin, 1991], for the pairs of spaces $(\boldsymbol{V}, Q)$: there exist constants β_d and β_s such that

$$\inf_{q \in Q} \sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\operatorname{div}} \|q\|} \geq \beta_d > 0, \tag{21}$$

$$\inf_{q \in Q} \sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1 \|q\|} \geq \beta_s > 0, \tag{22}$$

where the norms $\|\cdot\|_{\operatorname{div}}$, $\|\cdot\|_1$ and $\|\cdot\|$ denote the standard $H(\operatorname{div})$, H^1 and L^2 norms and $(\cdot,\cdot)$ is the L^2-inner product.

Examples in poromechanics: 2-field formulation

The two-field formulation of the quasi-static Biot's consolidation model after semidiscretization in time by the implicit Euler method, see, e.g.,

[Lee, Mardal, Winther 2017],

[Adler, Gaspar, Hu, Rodrigo, Zikatanov, 2018]

reads: find $(\boldsymbol{u}, p_F) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$ s.t.

$$
\begin{aligned}
(\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})) + \lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v}) - \alpha(p_F, \operatorname{div}\boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\\
-\alpha(\operatorname{div}\boldsymbol{u}, q_F) - c_0(p_F, q_F) - (\kappa \nabla p_F, \nabla q_F) &= (g, q_F), \qquad \forall q \in H_0^1(\Omega),
\end{aligned}
$$

where

- $\lambda \geq 0$ is a scaled Lamé coefficient,
- $c_0 \geq 0$ a storage coefficient,
- κ the (scaled) hydraulic conductivity,
- α the (scaled) Biot-Willis coefficient.

2-field formulation: bilinear forms and norms

The bilinear forms defining $\mathcal{A}((\cdot;\cdot),(\cdot;\cdot))$ are given by

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) &:= (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v})) + \lambda(\mathrm{div}\boldsymbol{u},\mathrm{div}\boldsymbol{v}), \qquad \forall \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}, \\ b(\boldsymbol{v},q_F) &:= -\alpha(\mathrm{div}\boldsymbol{v},q_F), \qquad \forall \boldsymbol{v}\in\boldsymbol{V}, \forall q_F\in Q, \\ c(p_F,q_F) &:= c_0(p_F,q_F) + (\kappa\nabla p_F,\nabla q_F), \qquad \forall p_F,q_F\in Q, \end{aligned}$$

where $Q := H_0^1(\Omega)$, $\boldsymbol{V} := \boldsymbol{H}_0^1(\Omega)$. We define $|\cdot|_Q$, $|\cdot|_V$ to be

$$\begin{aligned} |q_F|_Q^2 &:= \eta(q_F,q_F), \qquad \forall q_F\in Q, \\ |\boldsymbol{v}|_V^2 &:= (\varepsilon(\boldsymbol{v}),\varepsilon(\boldsymbol{v})) + \lambda(\mathrm{div}\boldsymbol{v},\mathrm{div}\boldsymbol{v}), \qquad \forall \boldsymbol{v}\in\boldsymbol{V}, \end{aligned}$$

where the parameter $\eta > 0$ is to be determined later. As before, $a(\boldsymbol{v},\boldsymbol{v}) \geq |\boldsymbol{v}|_V^2$ for all $\boldsymbol{v}\in\boldsymbol{V}$, that is, (17) is satisfied with $\underline{C}_a = 1$.

2-field formulation: small inf-sup condition

Obviously, there holds

$$\langle B\boldsymbol{v}, \bar{Q}^{-1} B\boldsymbol{v}\rangle_{Q'\times Q} \leq \frac{\alpha^2}{\eta}(\mathrm{div}\boldsymbol{v}, \mathrm{div}\boldsymbol{v}),$$

where $B : \boldsymbol{V} \to Q'$, $B := -\alpha\mathrm{div}$. Therefore, we obtain

$$\begin{aligned} \|\boldsymbol{v}\|_V^2 &= (\varepsilon(\boldsymbol{v}), \varepsilon(\boldsymbol{v})) + \lambda(\mathrm{div}\boldsymbol{v}, \mathrm{div}\boldsymbol{v}) + \langle B\boldsymbol{v}, \bar{Q}^{-1} B\boldsymbol{v}\rangle_{Q'\times Q} \\ &\leq (\varepsilon(\boldsymbol{v}), \varepsilon(\boldsymbol{v})) + \left(\lambda + \tfrac{\alpha^2}{\eta}\right)(\mathrm{div}\boldsymbol{v}, \mathrm{div}\boldsymbol{v}) \leq \left(1 + \lambda + \tfrac{\alpha^2}{\eta}\right)\|\boldsymbol{v}\|_1^2. \end{aligned}$$

We choose $\boldsymbol{v}_0$ such that $-\mathrm{div}\boldsymbol{v}_0 = \frac{1}{\sqrt{1+\lambda}} q_F$ and use (22) to obtain $\|\boldsymbol{v}_0\|_1 \leq \frac{1}{\beta_s}\frac{1}{\sqrt{1+\lambda}}\|q_F\|$, and finally

$$\begin{aligned} \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(\boldsymbol{v}, q_F)}{\|\boldsymbol{v}\|_V} &\geq \frac{b(\boldsymbol{v}_0, q_F)}{\|\boldsymbol{v}_0\|_V} = \frac{\frac{\alpha}{\sqrt{1+\lambda}}\|q_F\|^2}{\|\boldsymbol{v}_0\|_V} \geq \frac{\frac{\alpha}{\sqrt{1+\lambda}}}{\sqrt{\left(1 + \lambda + \frac{\alpha^2}{\eta}\right)}}\frac{\|q_F\|^2}{\|\boldsymbol{v}_0\|_1} \\ &\geq \frac{\beta_s \alpha}{\sqrt{\left(1 + \lambda + \frac{\alpha^2}{\eta}\right)}}\frac{\|q_F\|^2}{\|q_F\|} = \frac{\beta_s \alpha}{\sqrt{\left(1 + \lambda + \frac{\alpha^2}{\eta}\right)}}\frac{1}{\sqrt{\eta}}|q_F|_Q. \end{aligned}$$

2-field formulation: operator preconditioner

For $\eta := \frac{\alpha^2}{(1+\lambda)} > 0$ the right-hand side of the previous inequality is bounded from below by $\frac{\beta_s}{\sqrt{2}}|q_F|_Q$, which shows (18) with $\underline{\beta} = \frac{1}{\sqrt{2}}\beta_s$.

Hence we obtain the following norm-equivalent operator preconditioner:

$$\mathcal{B} := \begin{bmatrix} (-\mathrm{div}\varepsilon - (1+\lambda)\nabla\mathrm{div})^{-1} & \\ & \left(\left(c_0 + \alpha^2/(1+\lambda)\right) I - \mathrm{div}\kappa\nabla\right)^{-1} \end{bmatrix}$$

By introducing $p_S = -\lambda\mathrm{div}\boldsymbol{u}$ and substituting

- $\alpha p_F \to p_F$,
- $c_0\alpha^{-2} \to c_0$,
- $\kappa\alpha^{-2} \to \kappa$,
- $\alpha^{-1}g \to g$

in the *two-field formulation* we obtain the following three-field variational formulation of Biot's model, see [Lee, Mardal, Winther 2017]:

Solid-pressure-based 3-field formulation: bilinear forms

$$
\begin{aligned}
(\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})) - (p_S + p_F, \operatorname{div} \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\\
-(\operatorname{div} \boldsymbol{u}, q_S) - \lambda^{-1}(p_S, q_S) &= 0, \qquad \forall q_S \in L_0^2(\Omega),\\
-(\operatorname{div} \boldsymbol{u}, q_F) - c_0(p_F, q_F) - (\kappa \nabla p_F, \nabla q_F) &= (g, q_F), \qquad \forall q_F \in H_0^1(\Omega).
\end{aligned}
$$

The bilinear forms that determine $\mathcal{A}((\cdot;\cdot),(\cdot;\cdot))$ are

$$
\begin{aligned}
&a(\boldsymbol{u}, \boldsymbol{v}) := (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})), \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},\\
&b(\boldsymbol{v}, \boldsymbol{q}) := -(\operatorname{div} \boldsymbol{v}, q_S) - (\operatorname{div} \boldsymbol{v}, q_F), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \forall \boldsymbol{q} \in \boldsymbol{Q},\\
&c(\boldsymbol{p}, \boldsymbol{q}) := \lambda^{-1}(p_S, q_S) + c_0(p_F, q_F) + (\kappa \nabla p_F, \nabla q_F), \qquad \forall \boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{Q},
\end{aligned}
$$

where $\boldsymbol{V} = \boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{Q} = L_0^2(\Omega) \times H_0^1(\Omega)$ and $\boldsymbol{p} = (p_S; p_F)$, $\boldsymbol{q} = (q_S; q_F)$. Then the operator B is given by

$$
B := \begin{pmatrix} -\operatorname{div} \\ -\operatorname{div} \end{pmatrix}.
$$

Solid-pressure-based 3-field formulation: norms

We define $|\cdot|_Q$, $|\cdot|_V$ to be

$$|\boldsymbol{q}|_Q^2 := \left(\begin{pmatrix} I & I \\ I & I \end{pmatrix}\begin{pmatrix} q_S \\ q_{F,0} \end{pmatrix}, \begin{pmatrix} q_S \\ q_{F,0} \end{pmatrix}\right) = \|q_S + q_{F,0}\|^2, \qquad \forall \boldsymbol{q} \in \boldsymbol{Q},$$

$$|\boldsymbol{v}|_V^2 := (\varepsilon(\boldsymbol{v}), \varepsilon(\boldsymbol{v})), \qquad \forall \boldsymbol{v} \in \boldsymbol{V},$$

where $q_{F,0} := P_0 q_F$ and P_0 is the L^2 projection from $L^2(\Omega)$ to $L_0^2(\Omega)$. Then

$$\begin{aligned}\|\boldsymbol{q}\|_Q^2 &= \left(\begin{pmatrix} I & I \\ I & I \end{pmatrix}\begin{pmatrix} q_S \\ q_{F,0} \end{pmatrix}, \begin{pmatrix} q_S \\ q_{F,0} \end{pmatrix}\right) + \left(\begin{pmatrix} \lambda^{-1} I & 0 \\ 0 & c_0 I - \mathrm{div}\kappa\nabla \end{pmatrix}\begin{pmatrix} q_S \\ q_F \end{pmatrix}, \begin{pmatrix} q_S \\ q_F \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} (1+\lambda^{-1}) I & P_0 \\ P_0 & P_0 + c_0 I - \mathrm{div}\kappa\nabla \end{pmatrix}\begin{pmatrix} q_S \\ q_F \end{pmatrix}, \begin{pmatrix} q_S \\ q_F \end{pmatrix}\right) = (\bar{Q}\boldsymbol{q}, \boldsymbol{q}).\end{aligned}$$

As in the previous examples, (17) is satisfied with $\underline{C}_a = 1$.

Solid-pressure-based 3-field formulation: small inf-sup cond.

Next, we choose $\boldsymbol{v}_0$ such that $-\operatorname{div}\boldsymbol{v}_0 = q_S + q_{F,0}$ for which we have $\|\boldsymbol{v}_0\|_1 \le \beta_s^{-1}\|q_S + q_{F,0}\|$.

Then $b(\boldsymbol{v}_0, \boldsymbol{q}) = \|q_S + q_{F,0}\|^2 = |\boldsymbol{q}|_Q^2$ and

$$\begin{aligned}
\|\boldsymbol{v}_0\|_V^2 &= (\varepsilon(\boldsymbol{v}_0), \varepsilon(\boldsymbol{v}_0)) + (\bar{Q}^{-1}B\boldsymbol{v}_0, B\boldsymbol{v}_0) \\
&= (\varepsilon(\boldsymbol{v}_0), \varepsilon(\boldsymbol{v}_0)) + \left(\bar{Q}^{-1}\begin{pmatrix}-\operatorname{div}\boldsymbol{v}_0\\ -\operatorname{div}\boldsymbol{v}_0\end{pmatrix}, \begin{pmatrix}-\operatorname{div}\boldsymbol{v}_0\\ -\operatorname{div}\boldsymbol{v}_0\end{pmatrix}\right) \\
&\le \|\boldsymbol{v}_0\|_1^2 + \frac{1}{4}\left(\bar{Q}^{-1}\begin{pmatrix}I & P_0\\ P_0 & P_0\end{pmatrix}\begin{pmatrix}\operatorname{div}\boldsymbol{v}_0\\ \operatorname{div}\boldsymbol{v}_0\end{pmatrix}, \begin{pmatrix}I & P_0\\ P_0 & P_0\end{pmatrix}\begin{pmatrix}\operatorname{div}\boldsymbol{v}_0\\ \operatorname{div}\boldsymbol{v}_0\end{pmatrix}\right) \\
&\le \|\boldsymbol{v}_0\|_1^2 + \frac{1}{4}\left(\begin{pmatrix}I & P_0\\ P_0 & P_0\end{pmatrix}\begin{pmatrix}\operatorname{div}\boldsymbol{v}_0\\ \operatorname{div}\boldsymbol{v}_0\end{pmatrix}, \begin{pmatrix}\operatorname{div}\boldsymbol{v}_0\\ \operatorname{div}\boldsymbol{v}_0\end{pmatrix}\right) \\
&= \|\boldsymbol{v}_0\|_1^2 + (\operatorname{div}\boldsymbol{v}_0, \operatorname{div}\boldsymbol{v}_0) \\
&\le \beta_s^{-2}\|q_S + q_{F,0}\|^2 + \|q_S + q_{F,0}\|^2 = (\beta_s^{-2} + 1)|\boldsymbol{q}|_Q^2.
\end{aligned}$$

Solid-pressure-based 3-field formulation: preconditioner

Now (18) follows:

$$\sup_{\boldsymbol{v}\in V} \frac{b(\boldsymbol{v},\boldsymbol{q})}{\|\boldsymbol{v}\|_V} \geq \frac{b(\boldsymbol{v}_0,\boldsymbol{q})}{\|\boldsymbol{v}_0\|_V} \geq \frac{|\boldsymbol{q}|_Q^2}{\sqrt{(\beta_s^{-2}+1)}|\boldsymbol{q}|_Q} =: \underline{\beta}|\boldsymbol{q}|_Q, \quad \forall \boldsymbol{q} \in \boldsymbol{Q}.$$

Using the fitted norms for the constructions of a norm-equivalent preconditioner results in

$$\mathcal{B} := \begin{bmatrix} (-\mathrm{div}\varepsilon)^{-1} & \\ & \begin{pmatrix} (1+\lambda^{-1})I & P_0 \\ P_0 & P_0 + c_0 I - \mathrm{div}\kappa\nabla \end{pmatrix}^{-1} \end{bmatrix}.$$

In [Lee, Mardal, Winther 2017], the authors showed that the solid-pressure-based three-field formulation is not stable under the Q-seminorm defined by $|\boldsymbol{q}|_Q^2 = \|p_S\|^2 + \|p_F\|^2$.

Total-pressure-based 3-field formulation: bilinear forms

By introducing the total pressure $p_T = p_S + p_F$ in the previous example, another discrete in time three-field formulation of the quasi-static Biot's consolidation model, see [Lee, Mardal, Winther 2017], is obtained:

$$
\begin{aligned}
(\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})) - (p_T, \operatorname{div}\boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\
-(\operatorname{div}\boldsymbol{u}, q_T) - (\lambda^{-1} p_T, q_T) + (\alpha\lambda^{-1} p_F, q_T) &= 0, \quad \forall q_T \in L^2(\Omega), \\
(\alpha\lambda^{-1} p_T, q_F) - ((\alpha^2\lambda^{-1} + c_0) p_F, q_F) & \\
-(\kappa\nabla p_F, \nabla q_F) &= (g, q_F), \quad \forall q_F \in H_0^1(\Omega).
\end{aligned}
$$

Here, $\mathcal{A}((\cdot;\cdot),(\cdot;\cdot))$ is constructed from

$$
\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) :=& (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})), \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \\
b(\boldsymbol{v}, \boldsymbol{q}) :=& -(\operatorname{div}\boldsymbol{v}, q_T), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \forall \boldsymbol{q} \in \boldsymbol{Q}, \\
c(\boldsymbol{p}, \boldsymbol{q}) :=& (\lambda^{-1} p_T, q_T) - (\alpha\lambda^{-1} p_F, q_T) - (\alpha\lambda^{-1} p_T, q_F) \\
& + ((\alpha^2\lambda^{-1} + c_0) p_F, q_F) + (\kappa\nabla p_F, \nabla q_F), \qquad \forall \boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{Q},
\end{aligned}
$$

where $\boldsymbol{V} = \boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{Q} = L^2(\Omega) \times H_0^1(\Omega)$ and $\boldsymbol{p} = (p_T; p_F)$, $\boldsymbol{q} = (q_T; q_F)$.

Total-pressure-based 3-field formulation: norms & small inf-sup

Obviously, the operator $B : \boldsymbol{V} \to \boldsymbol{Q}'$ is defined by $B := \begin{pmatrix} -\mathrm{div} \\ 0 \end{pmatrix}$.

We next set

$$|\boldsymbol{q}|_Q^2 := (q_{T,0}, q_{T,0}), \qquad \forall \boldsymbol{q} \in \boldsymbol{Q},$$
$$|\boldsymbol{v}|_V^2 := (\varepsilon(\boldsymbol{v}), \varepsilon(\boldsymbol{v})), \qquad \forall \boldsymbol{v} \in \boldsymbol{V},$$

where $q_{T,0} := P_0 q_T$ is the L^2 projection of $q_T \in L^2(\Omega)$ to $L^2_0(\Omega)$.

Using similar arguments as in the previous examples, we obtain for $\boldsymbol{v}_0$ with $-\mathrm{div}\, \boldsymbol{v}_0 = q_{T,0}$ for which we have $\|\boldsymbol{v}_0\|_1 \le \beta_s^{-1} \|q_{T,0}\|$:

$$\begin{aligned}\|\boldsymbol{v}_0\|_V^2 &= (\varepsilon(\boldsymbol{v}_0), \varepsilon(\boldsymbol{v}_0)) + \langle B\boldsymbol{v}_0, \bar{Q}^{-1} B\boldsymbol{v}_0 \rangle \le (\varepsilon(\boldsymbol{v}_0), \varepsilon(\boldsymbol{v}_0)) + (\mathrm{div}\, \boldsymbol{v}_0, \mathrm{div}\, \boldsymbol{v}_0) \\ &\le 2\|\boldsymbol{v}_0\|_1^2 \le 2\beta_s^{-2} |\boldsymbol{q}|_Q^2.\end{aligned}$$

Again, (17) is satisfied with $\underline{C}_a = 1$ while (18) follows from

$$\sup_{\boldsymbol{v} \in V} \frac{b(\boldsymbol{v}, \boldsymbol{q})}{\|\boldsymbol{v}\|_V} \ge \frac{b(\boldsymbol{v}_0, \boldsymbol{q})}{\|\boldsymbol{v}_0\|_V} \ge \beta_s \frac{|\boldsymbol{q}|_Q^2}{|\boldsymbol{q}|_Q} =: \underline{\beta} |\boldsymbol{q}|_Q, \qquad \forall \boldsymbol{q} \in \boldsymbol{Q}.$$

Total-pressure-based 3-field formulation: preconditioner

Thus, the fitted norms generate the norm-equivalent preconditioner

$$\mathcal{B} := \begin{bmatrix} (-\mathrm{div}\varepsilon)^{-1} & \\ & \begin{pmatrix} \lambda^{-1} I + P_0 & -\alpha\lambda^{-1} I \\ -\alpha\lambda^{-1} I & \alpha^2\lambda^{-1} I + c_0 I - \mathrm{div}\kappa\nabla \end{pmatrix}^{-1} \end{bmatrix}.$$

Remark

The arguments presented above are valid also for a vanishing storage coefficient, i.e., $c_0 = 0$. Moreover, this analysis shows how solid- and total-pressure formulation are related to each other.

In fact, by the transformation

$$\begin{pmatrix} p_T \\ p_F \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} p_S \\ p_F \end{pmatrix} \text{ or, equivalently, } \begin{pmatrix} p_S \\ p_F \end{pmatrix} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} p_T \\ p_F \end{pmatrix},$$

we can derive stability and preconditioners for the two formulations from each other.

Total-pressure-based 3-field formulation: preconditioner

Note that for $c_0 \geq \alpha^2\lambda^{-1}$, as considered in [Lee, Mardal, Winther 2017], it is easy to show that

$$\|\boldsymbol{q}\|_Q^2 \eqsim \left((q_{T,0}, q_{T,0}) + (\lambda^{-1} q_T, q_T) + (\alpha^2\lambda^{-1} q_F, q_F) + (\kappa\nabla q_F, \nabla q_F)\right)$$

from which one obtains the stability result and the norm-equivalent preconditioner presented in [Lee, Mardal, Winther 2017], i.e.,

$$\mathcal{B}_0 := \begin{bmatrix} (-\mathrm{div}\varepsilon)^{-1} & \\ & \begin{pmatrix} \lambda^{-1} I + P_0 & \\ & \alpha^2\lambda^{-1} I - \mathrm{div}\kappa\nabla \end{pmatrix}^{-1} \end{bmatrix}.$$

Conclusion

We have proposed a new abstract framework for the stability analysis of perturbed saddle-point problems (PSPPs) with *arbitrary large perturbations* based on a concept of norm fitting and a small inf-sup type condition which

- is applicable to many PDE-based models, including vector Laplacian and various formulations of Biot's model,
- can be used to prove the uniform well-posedness of PSPPs in proper parameter-dependent norms,
- governs the construction of norm-equivalent preconditioners and optimal splitting schemes.

The framework [H., Kraus, Lymbery, Philo, 2021] also applies to discrete models, see e.g., [H., Kraus, Kuchta, Lymbery, Mardal, Rognes, 2021].

THANK YOU FOR YOUR ATTENTION!