Name: KEY ______________________ *Section:* 1&2

No Calculators.
Closed book and notes.
Write your final answers in the box if provided.
You may use the back of the pages.
You may not discuss with anyone not in your section until 5:00pm of test date.

MATH 353, ORD & PRTL DIFF EQUATIONS-EXAM 2

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	

1. Solve the eigenvalue problem (find all eigenvalues and the corresponding eigenfunctions)

$$\psi'' + \lambda\psi = 0, \quad 0 \le x \le 1,$$

with boundary conditions $\psi'(0) = 0$ and $\psi(1) = 0$. When determining the eigenvalues, examine the cases $\lambda = 0$ and $\lambda \neq 0$ separately.

$\underline{\lambda = 0}$ $\quad \psi'' = 0 \quad \psi = Ax + B \quad \psi'(0) = A = 0$

$\psi(1) = A + B = 0$

So $A = B = 0$

λ is not an eigenvalue

$\lambda \neq 0 \quad \psi = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$\psi' = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + \sqrt{\lambda}B\cos\sqrt{\lambda}x$

$\psi'(0) = \sqrt{\lambda}B = 0$, $B = 0$, $\psi = A\cos\sqrt{\lambda}x$

$\psi(1) = A\cos\sqrt{\lambda} = 0$, $\sqrt{\lambda_n} = n\pi - \frac{\pi}{2} \quad n = 1, 2, 3 \cdots$

$\lambda_n = (n - \frac{1}{2})^2\pi^2 \quad n = 1, 2, 3 \cdots$

$\psi_n = A_n\cos[(n - \frac{1}{2})\pi x] \quad A_n \neq 0$ arbitrary

2. In the following table, indicate whether the operator is or is not selfadjoint. An x in the slot of selafadjoint column means the operator is selfadjoint. An x in the non-selfadjoint column means it is not.

   Clarification: A function (as with $\sin x$ and $\cos x$ in the table) acting as an operator function, simply multiplies the function it is acting on.

   There is a penalty for a wrong answer and for marking both slots.

operator	boundary conditions	selfadjoint	non-selfadjoint
$\frac{d^2}{dx^2} + 3\frac{d}{dx}$	periodic		X
$\frac{d^2}{dx^2}$	Dirichlet [H]	X	
$\frac{d^2}{dx^2} + \sin x$	Neumann [H]	X	
$\cos x$	none, $x \in \mathbb{R}$	X	
$i\frac{d}{dx}$	none, $x \in \mathbb{R}$	X	

3. The inner product in the vector space of real square-integrable functions defined on the interval $0 < x < 1$ is defined by

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Let the functions $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$, $\cdots$ be an orthogonal (not necessarily orthonormal) basis of this space and let $f(x)$ be a known square-integrable function, defined on the interval $0 < x < 1$.

(a) Derive a formula for the coefficient a_n of the expansion

$$f(x) = a_1\psi_1(x) + a_2\psi_2(x) + a_3\psi_3(x) + \cdots, \qquad 0 < x < 1,$$

(no credit for writing the formula from memory)

Take inner product with $\psi_k(x)$

$$(f, \psi_k) = a_1(\psi_1, \psi_k) + a_2(\psi_2, \psi_k) + \cdots$$

We know $(\psi_m, \psi_k) = 0$, if $m \neq k$

$$(f, \psi_k) = a_k(\psi_k, \psi_k), \quad \boxed{a_k = \frac{(f, \psi_k)}{(\psi_k, \psi_k)}}$$

(b) Given a second real square-integrable function, defined on the interval $0 < x < 1$

$$g(x) = b_1\psi_1(x) + b_2\psi_2(x) + b_3\psi_3(x) + \cdots, \qquad 0 < x < 1,$$

and now assuming that the basis is ORTHONORMAL, prove that

$$(f, g) = a_1b_1 + a_2b_2 + a_3b_3 + \cdots.$$

$$(f, g) = (a_1\psi_1 + a_2\psi_2 + \cdots, b_1\psi_1 + b_2\psi_2 + \cdots)$$

All cross-terms (ψ_m, ψ_n) with $m \neq n$ are zero

Thus $$(f, g) = a_1b_1\underbrace{(\psi_1, \psi_1)}_{=1} + a_2b_2\underbrace{(\psi_2, \psi_2)}_{=1} + \cdots$$

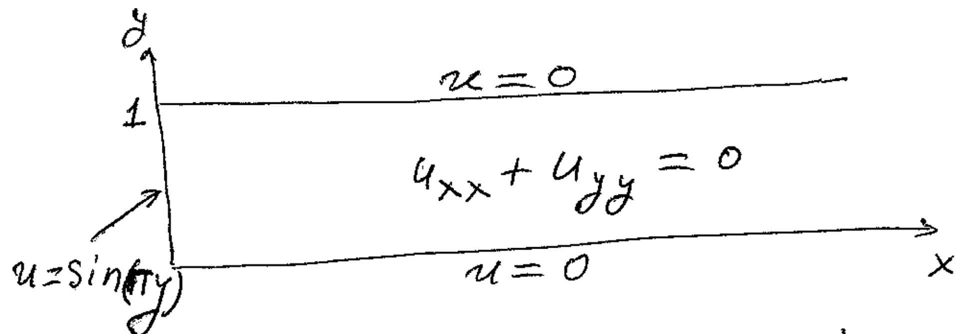
by orthonormality

4. Solve the following boundary value problem for Laplace's equation without using a series expansion.

$u_{xx} + u_{yy} = 0$ in the region of the x, y plane in which $0 < x < \infty$ and $0 < y < 1$,

Boundary conditions:

$$\begin{cases} u = 0 \text{ when } x > 0,\ y = 0, \\ u = 0 \text{ when } x > 0,\ y = 1, \\ u(0, y) = \sin(\pi y) \text{ when } x = 0,\ 0 < y < 1, \\ u \text{ remains bounded as } x \text{ tends to infinity.} \end{cases}$$



$\sin(\pi y)$ is an eigenfunction ~~eigenvector~~ of the operator $\mathbb{L} = -\frac{d^2}{dy^2}$

Look for solution $u = b(x)\sin(\pi y)$

Insert into PDE

$$b''(x)\sin(\pi y) - \pi^2 b(x)\sin(\pi y) = 0$$

$$\left(b''(x) - \pi^2 b(x)\right)\sin(\pi y) = 0 \qquad \forall x$$

$$b'' - \pi^2 b = 0 \qquad b(x) = Ae^{-\pi x} + Be^{\pi x}$$

By the boundedness condition $B = 0$

BC: $u(0,y) = 1$ thus $b(0) = 1$, thus $A = 1$

$$u(x) = e^{-\pi x}\sin(\pi y)$$

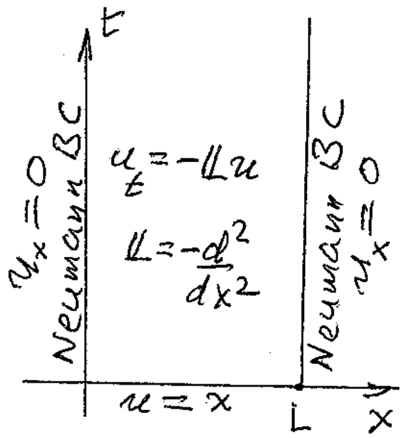
5. (a) Solve the following initial boundary value problem for $u(x,t)$ when $m = 0$.

$$u_t = u_{xx} + m\cos\frac{\pi x}{L}, \quad 0 < x < L, \quad t > 0,$$

$$\text{BC:} \quad u_x(0,t) = 0, \quad u_x(L,t) = 0,$$

$$\text{IC:} \quad u(x,0) = x,$$

NOTE: You may use the appropriate basis if you remember it without deriving it. Remember to check whether $\lambda = 0$ is an eigenvalue. Remember the condition $m = 0$



Will use eigenfunction expansion.
e-values, e-functions of operator $\mathbb{L}$ with Neumann BC:
e-values: $0, \left(\frac{\pi}{L}\right)^2, \dots, \left(\frac{n\pi}{L}\right)^2, \dots$
e-functions: $1, \cos\frac{\pi x}{L}, \dots, \cos\frac{n\pi x}{L}, \dots$
Seek solution of form $b(t)\psi(x)$ where $\psi(x)$ is an e-function of $\mathbb{L}$, thus, $\mathbb{L}\psi = \lambda\psi$.

$m=0$

PDE: $\dot{b}\psi = -b\mathbb{L}\psi = -\lambda b\psi$, $\dot{b}\cancel{\psi} = -\lambda b\cancel{\psi}$ cancel at x with $\psi(x) \neq 0$

ODE: $\dot{b} = -\lambda b$, $b(t) = b(0)e^{-\lambda t}$. Calculate $b(0)$: $(x,\psi) = b(0)(\psi,\psi)$

$n \neq 0$: $b_n(0) = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx$

$b_0(0) = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$

Superposition of solutions

$u(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty} b_n(0)\, e^{-\left(\frac{n}{\pi L}\right)^2 t}\cos\frac{n\pi x}{L}$

(b) How will the solution change if $m \neq 0$?

$m \neq 0$

The only contribution affected is the one for $n=1$.

ODE: $\dot{b}_1 = -\lambda_1 b_1 + m$. Solution

$b_1 = \frac{m}{\lambda_1} + \left(b_1(0) - \frac{m}{\lambda_1}\right)e^{-\lambda_1 t}$

(c) What is the limit of the solution as t tends to positive infinity in (a) and (b) above

(a) $\lim_{t\to+\infty} u(x,t) = \frac{L}{2}$

(b) $\lim_{t\to+\infty} u(x,t) = \frac{L}{2} + \frac{m}{\lambda_1}\cos\frac{\pi x}{L}$

6. (a) Calculate the Fourier series of the function $f(x)$ that equals x^2 when $-\pi \leq x \leq \pi$ and is repeated periodically with period 2π over the x axis.

$$x^2 = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(mx) + \sum_{m=1}^{\infty} b_m \sin(mx)$$

$$\int_{-\pi}^{\pi} x^2 dx = \frac{a_0}{2} \cdot 2\pi \qquad a_0 = \frac{2\pi^2}{3}$$

$$k=1,2,3\ldots \quad \int_{-\pi}^{\pi} x^2 \cos kx \, dx = a_k \int_{-\pi}^{\pi} \cos^2 kx \, dx \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos kx \, dx$$

Integration by parts: $a_k = \frac{4}{k^2}\cos(k\pi) = \frac{4(-1)^k}{k^2}$, $k=1,2,3\ldots$

$b_k = 0$, x^2 is even

So $$x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{k^2}, \qquad -\pi < x < \pi$$

(b) Use the result of (a) to evaluate the series

$$A = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$$

and the series

$$B = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

To calculate A, let $x=0$ in the series representation of x^2

$$0 = \frac{\pi^2}{3} - 4A \qquad A = \frac{\pi^2}{12}$$

To calculate B, let $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4B \qquad B = \frac{\pi^2}{6}$$