ORD & PRTL DIFF EQUATIONS-MATH 353-SPRING 2014-EXAM 2

Name KEY

Section ____________________

Thursday, April 17, 2014.

No Calculators. No cellphones.
Closed book and notes.
You may use the back of the pages.
Students' "no assistance" pledge

DO FIVE OF THE SIX PROBLEMS
CROSS OUT THE GRADE BOX OF THE PROBLEM NOT DONE

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	

PLACE YOUR FINAL ANSWERS IN A BOX

1. Find all the nonzero solutions of the following ODE boundary value problem and the corresponding values of the parameter $a \geq 0$.

$$y_{xx} + a^2 y = 0, \quad 0 < x < \pi,$$

$$y(0) = 0, \quad y_x(\pi) = 0$$

<u>Case 1</u> $a = 0$

$y'' = 0$ $\quad y = c_1 x + c_2$

$y(0) = 0$ $\quad c_2 = 0$ $\quad$ No nonzero solution

$y_x(\pi) = 0$ $\quad c_1 = 0$

<u>Case 2</u> $a \neq 0$

$y = c_1 \cos(ax) + c_2 \sin(ax)$

$y(0) = 0$: $\quad c_1 = 0$ $\quad$ so $y(x) = c_2 \sin(ax)$ $\quad c_2 \neq 0$

$y_x(\pi) = 0$; $\quad \cos(a\pi) = 0$ $\quad a\pi = n\pi - \frac{\pi}{2}$

$a_n = \left(n - \frac{1}{2}\right)$, $\quad n = 1, 2, \dots$

(all other integers n do not produce new solutions $y_n(x)$).

<u>Ans.</u> $a_n = n - \frac{1}{2}$ $\quad n = 1, 2, 3, \dots$

$y_n(x) = C_n \sin\left[\left(n - \frac{1}{2}\right) x\right]$, $\quad n = 1, 2, 3 \dots$

2. Find the solution to the Laplace equation

$$u_{xx} + u_{yy} = 0; \qquad 0 < x < \pi, \quad y > 0.$$

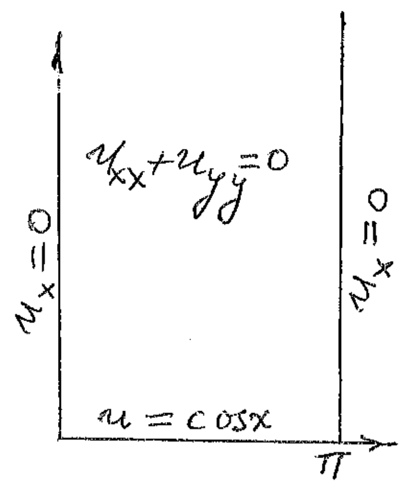
You are given the following conditions :

(a) $u_x = 0$ along the vertical half-line $x = 0,\ y > 0$ and along the vertical half-line $x = \pi,\ y > 0$.

(b) $u = \cos x$ on x axis $(0 < x < \pi)$.

(c) u remains bounded as y tends to infinity.

Hint: Can be done with little calculation. Begin by sketching the region of validity of the PDE.



We have zero Neumann bound. condition. The eigenfunctions ~~ease~~ that form a basis are

$1, \cos x, \cos 2x, \cos 3x, \ldots$

Only $\cos x$ appears in the input to the problem, that is in the initial condition, thus the solution is $u = b(y)\cos x$. Insert into PDE:

$$\begin{cases} b_{yy} - b = 0 \\ b(0) = 1 \end{cases}, \quad b(y) = c_1 e^{y} + c_2 e^{-y}$$

$c_1 = 0$ (requirement (c))

$b(0) = 1 \implies c_2 = 1$

Answer: $u(x,y) = e^{-y}\cos x$

3. Given two arbitrary functions $F(x)$ and $G(x)$,

(a) verify that the function,

$$u(x,t) = F(x-t) + G(x+t)$$

satisfies (is a solution of) the wave equation

$$u_{tt} - u_{xx} = 0.$$

$$u_{tt} = F''(x-t) + G''(x+t)$$
$$u_{xx} = F''(x-t) + G''(x+t)$$

subtract $\quad u_{tt} - u_{xx} = 0$

(b) determine the functions F and G, if you know in addition that

$$u(x,0) = \frac{1}{1+x^2} \quad \text{and} \quad u_t(x,0) = 0$$

$$u(x,0) = F(x) + G(x) = \frac{1}{x^2+1}$$
$$u_t(x,0) = -F'(x) + G'(x) = 0 \implies G = F + \text{const}$$

with trivial algebra: $F(x) = \frac{1}{2(x^2+1)} - \frac{c}{2}$

$$G(x) = \frac{1}{2(x^2+1)} + \frac{c}{2}$$

$$u(x,t) = \frac{1}{2}\,\frac{1}{(x-t)^2+1} + \frac{1}{2}\,\frac{1}{(x+t)^2+1}$$

(c) Graph $u(x)$ at $t = 5$. Describe in words how the wave evolves.

left-traveling wave $\qquad$ right-traveling wave

4. (a) Let $\mathbb{L}$ be a second order linear differential operator over an interval (a, b) with homogeneous boundary conditions. What does it mean to say that the operator $\mathbb{L}$ is selfadjoint?

For any two square-integrable, twice differentiable functions f and g on (a,b)

$$(\mathbb{L}f, g) = (f, \mathbb{L}g)$$

(b) Give an example of a second order linear differential operator over (a, b) with homogeneous Dirichlet boundary conditions, that is not selfadjoint. Explain why it is not selfadjoint.

$\mathbb{L} = \frac{d^2}{dx^2} + \frac{d}{dx}$ with zero Dirichlet conditions.

Selfadjointness of $\frac{d^2}{dx^2}$ is ruined by $\frac{d}{dx}$

$$\left(\frac{d}{dx} f, g\right) = -\left(f, \frac{d}{dx} g\right)$$

(c) True or False ? (circle correct, penalty for wrong answer)

In the following initial boundary value problem for the heat equation,

- $u_t = a^2 u_{xx}$ over the interval $0 < x < L$ and for $t > 0$,
- $u(0, t) = u(L, t) = 0$,

necessarily, $u(x, t)$ tends to zero as t tends to infinity.

Give a mathematics and a physics justification of your answer.

Math. explanation: $\lambda = 0$ is not an eigenvalue. The coefficients $b_n(t)$ of all $\psi_n(x)$ decay exponentially as $t \to \infty$

Phys. explanation: Temperature eventually equalizes at the zero boundary value. Heat is lost at the endpoints

(d) If in the previous problem the boundary condition at both endpoints is of Neumann type ($u_x = 0$), what will be the limit of $u(x, t)$ as t tends to infinity? Assume the initial condition $u(x, 0) = 2.5 + cos(153x)$.

$$\lim_{t \to \infty} u(x,t) = 2.5$$

5. (a) A given function $f(x)$ is defined on the interval $0 < x < L$ and belongs to the space L^2 of this interval. The function is represented as a series

$$f(x) = \sum_{m=1}^{\infty} c_m \sin \frac{m\pi x}{L}.$$

Give a step-by-step derivation of a formula for the coefficients with appropriate justifications.

multiply by $\sin \frac{n\pi x}{L}$ on both sides.
and integrate from 0 to L.

$$\int_0^L f(x) \sin \frac{n\pi x}{L} dx = \sum_{m=1}^{\infty} c_m \underbrace{\int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx}_{\text{equals } \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m. \end{cases}}$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

(b) Write down a linear PDE that is satisfied by the function $\sin x \cos(2y)$.

$$4u_{xx} - u_{yy} = 0$$

(c) Find the solution of the following boundary value problem WITHOUT DOING ANY CALCULATION.

- $u_{xx} + u_{yy} = 0$ inside a rectangle in the (x, y) plane,
- At every point of the boundary the derivative of u in the direction perpendicular to the boundary equals zero.

$u(x,y) = \text{constant}$

6. Find the solution to the following initial-boundary value problem

$$u_t = u_{xx} + f(x), \quad 0 < x < \pi, \quad t > 0$$
$$u_x(0,t) = 0, \quad u_x(\pi, t) = 0$$
$$u(x,0) = 0$$

$u(x,t) = \sum_n b_n(t)\psi_n(x)$ basis to be determined

Since $u(x,0)=0$, we have $\boxed{b_n(0)=0 \text{ for all } n}$

$f(x) = \sum_n c_n \psi_n(x)$

Insert series for $u(x,t)$ and $f(x)$ into PDE

(*) $\sum_n \dot{b}_n \psi_n = \sum_n b_n \psi_{n_{xx}} + \sum_n c_n \psi_n$

Choose basis by the property

(1) $-\psi_{xx} = \lambda \psi$ (Makes all series in (*) be series of ψ_n).

(2) BC $\psi_x(0)=0$ $\psi_x(\pi)=0$ (Neumann BC)

Solve this e-value problem to find:

$\psi_n(x) = \cos(nx)$ $n=0,1,2,\dots$ notice $\psi_0 = 1$

$\lambda_n = n^2$

$\psi_{n_{xx}} = -\lambda_n \psi_n$ → Insert into (*) and collect

$$\sum_{n=0}^{\infty} \dot{b}_n \psi_n = \sum_{n=0}^{\infty}(-\lambda_n b_n + c_n)\psi_n.$$

Balance coefficients.

$$\begin{cases} \dot{b}_n = -\lambda_n b_n + c_n & n=0,1,2,\dots \\ b_n(0) = 0 \end{cases}$$

Solution:

$b_0(t) = c_0 t$

$b_n(t) = \frac{c_n}{n^2}\left(1 - e^{-n^2 t}\right)$ $n=1,2,3\dots$

Form of answer: $u(x,t) = \sum b_n(t)\psi_n(x)$

Range of index: $n = 0, 1, 2 \dots$

Basis functions ψ_n: $\cos(nx)$

Eigenvalues λ_n: n^2

note $\psi_0(x) = 1$, $\lambda_0 = 0$

Coefficient initial values $b_n(0)$: All 0

Coefficients $b_n(t)$: $\begin{cases} \frac{c_n}{n^2}\left(1-e^{-n^2 t}\right) & n=1,2,\dots \\ c_0 t, & n=0 \end{cases}$

Find c_n:

$c_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ $\quad c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$