Math 353 Name:

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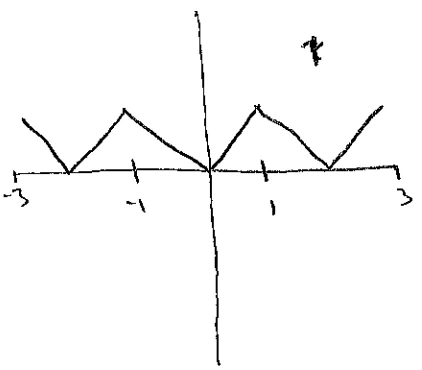
MIDTERM II Solutions

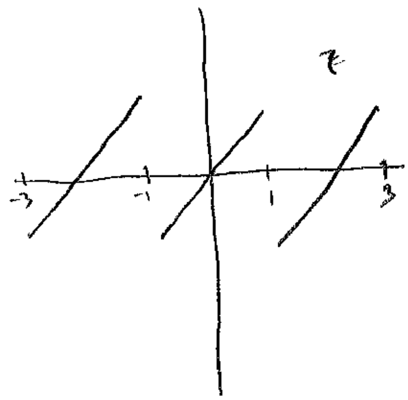
Directions: Read each question carefully and answer all parts. No calculators, notes, or devices other than your brain. <u>Show your work</u> - no credit will be given if you do not show your work. Good luck!

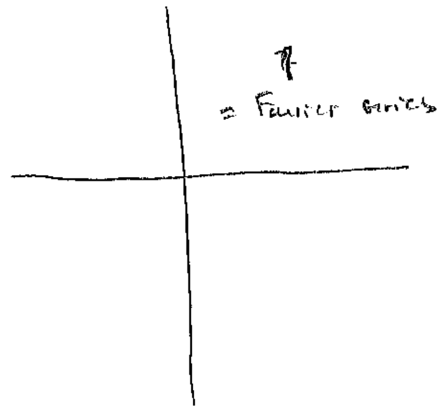
1. Draw the Fourier series, Fourier sine series, and Fourier cosine series of the following functions:

Fourier series | Fourier sine series | Fourier cosine series

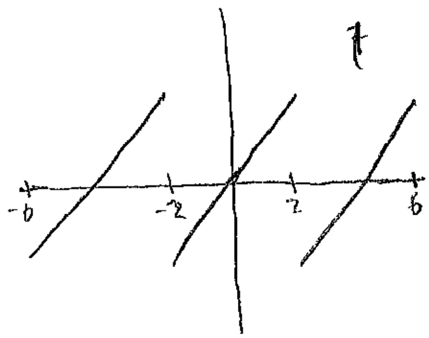
$f(x) = |x|$ $(L=1)$

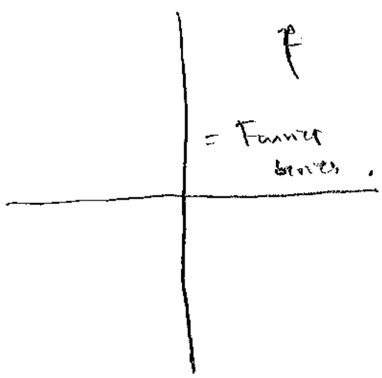


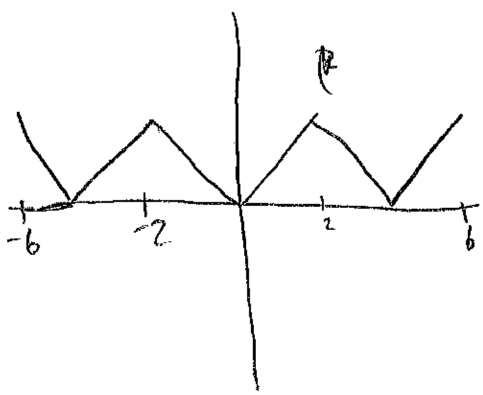




$f(x) = x$ $(L=2)$







Find a function $f(x)$ such that Fourier series of $f(x)$ = Fourier sine series of $f(x)$ = Fourier cosine series of $f(x)$.

$f(x) = 0$

2. Let L be a Sturm-Liouville operator. Consider the following eigenvalue problem:

$$L[y] = \lambda y \quad y'(0) = 0 \quad y'(2) = 0.$$

Let $\{\varphi_n(x)\}$ be the eigenfunctions with eigenvalues $\{\lambda_n\}$.

Define the inner product with respect to which the eigenfunctions are orthogonal:

$$\langle h_1(s), h_2(s) \rangle = \int_0^2 h_1(s) h_2(s)\, ds \qquad 1$$

Suppose $\langle \varphi_n, \varphi_n \rangle = n^2$ for all n. Expand $f(x)$ in a series of eigenfunctions $\{\varphi_n\}$.

$f(x) = \sum_n c_n \varphi_n(x)$; to find c_k, take inner product w/ φ_k:

$\langle f(x), \varphi_k(x) \rangle = \langle \sum_n c_n \varphi_n(x), \varphi_k(x) \rangle$; by orthoganility, $\langle \varphi_n, \varphi_k \rangle = 0$ for $n \neq k$;

$$\langle f, \varphi_k \rangle = \sum_n c_n \langle \varphi_n, \varphi_k \rangle$$

$$\langle f, \varphi_k \rangle = c_k \underbrace{\langle \varphi_k, \varphi_k \rangle}_{= k^2} \qquad 4$$

$$c_k = \frac{1}{k^2} \langle f, \varphi_k \rangle = \frac{1}{k^2} \int_0^2 f(x) \varphi_k(x)\, dx$$

Consider the eigenvalue problem above with $L = -\frac{d^2}{dx^2}$.

Is this a Sturm-Liouville problem? yes $(p(x) = 1,\ q(x) = 0,\ \alpha_1 = \beta_1 = 0,\ \alpha_2 = \beta_2 = 1,\ a = 2)$ 1

Find a set of eigenfunctions $\{\varphi_n\}$ satisfying $\langle \varphi_n, \varphi_n \rangle = n^2$ for all n.

Eigenvalues: $\lambda_n = \frac{n^2 \pi^2}{4}$ $(n = 0, 1, 2, \ldots)$ 2

Eigenfunctions: $\tilde{\varphi}_n(x) = \cos \frac{n\pi x}{2}$

Define $\varphi_n(x) = n \tilde{\varphi}_n(x) = n \cos \frac{n\pi x}{2}$ $(n > 0)$ (not possible for $\varphi_0(x)$)

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Then $\langle \varphi_n, \varphi_n \rangle = \int_0^2 \left(n \cos \frac{n\pi x}{2}\right)^2 dx = n^2 \int_0^2 \left(\cos \frac{n\pi x}{2}\right)^2 dx = n^2$ $(n > 0)$

3. Solve the following Schrödinger wave equation problem for $\Psi(x,t)$:

$$i\Psi_t = \Psi_{xx} \qquad \Psi(0,t) = 0 \qquad \Psi(\pi,t) = 0$$

$$\Psi_t(x,0) = f(x).$$

$\left(\text{Note: } \frac{1}{i} = \frac{i}{i^2} = -i\right)$

Assume $\Psi(x,t) = \sum_n b_n(t)\varphi_n(x)$; plugging in:

$$i\sum_n b_n'(t)\varphi_n(x) = \sum_n b_n(t)\varphi_n''(x)$$

Eigenvalue problem:

$$\varphi'' = -\lambda\varphi$$

$$\Psi(0,t) = 0 \Rightarrow \varphi(0) = 0$$

$$\Psi(\pi,t) = 0 \Rightarrow \varphi(\pi) = 0$$

Eigenvalues: $\lambda_n = n^2$ $(n = 1,2,3,\ldots)$

Eigenfcns: $\varphi_n(x) = \sin nx$

$$\langle h_1, h_2\rangle = \int_0^\pi h_1(s)h_2(s)\,ds$$

Replacing φ_n'' by $-\lambda_n\varphi_n$, eqn becomes:

$$i\sum_{n=1}^\infty b_n'(t)\varphi_n(x) = \sum_{n=1}^\infty b_n(t)(-\lambda_n\varphi_n(x))$$

$$\sum_{n=1}^\infty \left[i b_n'(t) + \lambda_n b_n(t)\right]\varphi_n(x) = 0$$

$$i b_n'(t) + \lambda_n b_n(t) = 0 \quad (n = 1,2,\ldots)$$

$$b_n'(t) + \frac{\lambda_n}{i} b_n(t) = 0$$

Integrating factor: $e^{\frac{\lambda_n}{i}t}$

Multiply by integrating factor:

$$\left[b_n(t)e^{\frac{\lambda_n}{i}t}\right]' = 0$$

$$b_n(t) = c_n e^{-\frac{\lambda_n}{i}t} = c_n e^{i\lambda_n t} = c_n e^{in^2 t}.$$

$$\Psi(x,t) = \sum_{n=1}^\infty c_n e^{in^2 t}\sin nx$$

$$\Psi_t(x,t) = \sum_{n=1}^\infty c_n i n^2 e^{in^2 t}\sin nx$$

$$\Psi_t(x,0) = \sum_{n=1}^\infty c_n i n^2 \sin nx = f(x).$$

to find c_k, take $\langle\,\rangle$ w/ φ_k:

$$\sum_{n=1}^\infty c_n i n^2 \underbrace{\langle \sin nx, \sin kx\rangle}_{\substack{=0 \ (n\neq k) \\ =\frac{\pi}{2} \ (n=k)}} = \langle f(x), \sin kx\rangle$$

$$c_k i k^2 \frac{\pi}{2} = \langle f, \varphi_k\rangle$$

$$c_k = \frac{2}{ik^2\pi}\int_0^\pi f(x)\varphi_k(x)\,dx$$

Why do we call this a wave equation rather than a heat equation?

The i in e^{in^2t} causes the solution to exhibit oscillatory "standing wave" behavior rather than diffusion behavior.

4. Solve the following wave equation problem for $u(x,t)$:

$$u_{tt} = u_{xx} + 9\pi^2 \cos 3\pi x \qquad u_x(0,t)=0 \quad u_x(1,t)=0$$

$$u(x,0)=0 \qquad u_t(x,0)=0.$$

Assume $u(x,t) = \sum_n b_n(t)\phi_n(x)$; plugging in:

$$\sum_n b_n''(t)\phi_n(x) = \sum_n b_n(t)\phi_n''(x) + 9\pi^2\cos 3\pi x$$

Eigenvalue problem:

$\phi'' = -\lambda\phi$

$u_x(0,t)=0 \Rightarrow \phi'(0)=0$

$u_x(1,t)=0 \Rightarrow \phi'(1)=0$

$$\boxed{\begin{aligned}&\lambda_n = n^2\pi^2 \quad (n=0,1,2,\dots)\\ &\phi_n(x) = \cos n\pi x\\ &\langle h_1,h_2\rangle = \int_0^1 h_1(s)\,h_2(s)\,ds\end{aligned}}$$

Expand $9\pi^2\cos 3\pi x$ in a series of eigenfunctions:

$$\boxed{\begin{aligned}&9\pi^2\cos 3\pi x = \sum_{n=0}^{\infty}\gamma_n(t)\cos n\pi x\\ &\text{By inspection,}\\ &\gamma_n(t) = \begin{cases}0 & n\neq 3\\ 9\pi^2 & n=3\end{cases}\end{aligned}}$$

Replacing ϕ_n'' by $-\lambda_n\phi_n$ and $9\pi^2\cos 3\pi x$ by its eigenfunction expansion:

$$\sum_{n=0}^{\infty} b_n''(t)\phi_n(x) = \sum_{n=0}^{\infty} b_n(t)\left(-\lambda_n\phi_n(x)\right) + \sum_{n=0}^{\infty}\gamma_n(t)\phi_n(x)$$

$$\sum_{n=0}^{\infty}\left[b_n'' + \lambda_n b_n - \gamma_n(t)\right]\phi_n(x) = 0$$

$$b_n'' + \lambda_n b_n = \gamma_n \qquad (n=0,1,2,3,\dots)$$

$\underline{n\neq 3}$

$b_n'' + \lambda_n b_n = 0$

$b_n(t) = c_{n1}\cos\sqrt{\lambda_n}\,t + c_{n2}\sin\sqrt{\lambda_n}\,t$

$$\left.\begin{aligned}&u(x,0)=0 \Rightarrow b_n(0)=0\\ &u_t(x,0)=0 \Rightarrow b_n'(0)=0\end{aligned}\right\}\ c_{n1}=c_{n2}=0$$

$b_n(t) = 0$

$\underline{n=3}$

$b_3'' + 9\pi^2 b_3 = 9\pi^2$

By inspection, a particular solution is $Y_p(t) = 1$

$b_3(t) = c_{31}\cos 3\pi t + c_{32}\sin 3\pi t + 1$

$b_3(0)=0 \Rightarrow c_{31}(1) + c_{32}(0) + 1 = 0 \Rightarrow c_{31} = -1$

$b_3'(0)=0$: $b_3'(t) = 3\pi\sin 3\pi t + 3\pi c_{32}\cos 3\pi t$

$$\left.b_3'(0) = 3\pi c_{32}(1) = 0\right\} \Rightarrow c_{32} = 0$$

$$u(x,t) = \left[-\cos 3\pi t + 1\right]\cos 3\pi x$$

5. Consider the following PDE problem for $u(x,t)$:

$$u_t = x u_{xx} + u_x + t x^3 \sin \pi x$$

$$u(0,t) + 3u_x(0,t) = 0 \qquad u(x,0) = 0$$

$$u(4,t) = 0.$$

Assuming the solution can be written in the form of an eigenfunction expansion, find the eigenvalue problem for the eigenfunctions. Do not solve.

Assume $u(x,t) = \sum_n b_n(t) \phi_n(x)$; plugging in:

$$\sum_n b_n'(t)\phi_n(x) = x \sum_n b_n(t)\phi_n''(x) + \sum_n b_n(t)\phi_n'(x) + t x^3 \sin \pi x$$

$$\sum_n b_n(t)\phi_n(x) = \sum_n b_n(t)\, x\phi_n''(x) + \sum_n b_n(t)\phi_n'(x) + t x^3 \sin \pi x \quad 2$$

$$\sum_n b_n(t)\phi_n(x) = \sum_n b_n(t)\left[x\phi_n''(x) + \phi_n'(x)\right] + t x^3 \sin \pi x$$

$$x\phi'' + \phi' = -\lambda \phi \quad 3$$

$$u(0,t) + 3u_x(0,t) = 0 \Rightarrow \phi(0) + 3\phi'(0) = 0$$

$$u(4,t) = 0 \Rightarrow \phi(4) = 0 \quad 3$$

Can this problem be solved using the eigenfunction expansion method? Explain why or why not and justify your answer.

yes $\Rightarrow$ because the eigenvalue problem for the $\{\phi_n\}$ is a Sturm-Liouville eigenvalue problem

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has this form:

$p(x) = -x$

$q(x) = 0$

$\alpha_1 = 1 \quad \alpha_2 = 3$

$\beta_1 = 1 \quad \beta_2 = 0 \quad a = 4$

general form for SL problem:

$$L = -p(x)\frac{d^2}{dx^2} - p'(x)\frac{d}{dx} + q(x)$$

$$L[y] = \lambda y$$

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0 \qquad \beta_1 y(a) + \beta_2 y'(a) = 0$$

6. Consider the following nonhomogeneous wave equation problem for $u(x,t)$:

$$\left.\begin{aligned} &u_{tt} = u_{xx} + F(x,t) \\ &u(x,0) = f(x) \qquad u(0,t) = T_1 \\ &u_t(x,0) = g(x) \qquad u(L,t) = T_2. \end{aligned}\right\} \quad (*)$$

This is a problem we cannot directly solve. However, the solution $u(x,t)$ can be written as $u(x,t) = w(x,t) + v(x,t) + z(x)$, where $w(x,t)$, $v(x,t)$, and $z(x)$ are solutions to problems that we <u>can</u> solve. Write down the problems that $w(x,t)$, $v(x,t)$, and $z(x)$ solve, and show that $u(x,t) = w(x,t) + v(x,t) + z(x)$ solves $(*)$.

Problem for $z(x)$:

$z''(x) = 0$

$z(0) = T_1$

$z(L) = T_2$

Problem for $w(x,t)$:

$w_{tt} = w_{xx}$

$w(x,0) = f(x) - z(x) \qquad w(0,t) = 0$

$w_t(x,0) = 0 \qquad w(L,t) = 0$

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Problem for $v(x,t)$:

$v_{tt} = v_{xx} + F(x,t)$

$v(x,0) = 0 \qquad v(0,t) = 0$

$v_t(x,0) = g(x) \qquad v(L,t) = 0$

Show $u(x,t) = w(x,t) + v(x,t) + z(x)$ solves $(*)$:

1) $u_{tt} = [w+v+z]_{tt} = \underbrace{w_{tt}}_{=w_{xx}} + \underbrace{v_{tt}}_{=v_{xx}+F(x,t)} + \underbrace{z_{tt}}_{=0} = w_{xx} + v_{xx} + F(x,t) = w_{xx} + v_{xx} + z_{xx} + F(x,t)$ (since $z''=0$)

$= [w+v+z]_{xx} + F(x,t) = u_{xx} + F(x,t)$ ✓

2) $u(0,t) = w(0,t) + v(0,t) + z(0) = 0 + 0 + T_1 = T_1$ ✓ 4

$u(L,t) = w(L,t) + v(L,t) + z(L) = 0 + 0 + T_2 = T_2$

$u(x,0) = w(x,0) + v(x,0) + z(x) = f(x) - z(x) + 0 + z(x) = f(x)$ ✓

$u_t(x,0) = w_t(x,0) + v_t(x,0) + z_t(x) = 0 + g(x) + 0 = g(x)$

Can $z(x)$ be thought of as the steady state solution to $(*)$? Explain your answer.

No - this is a wave problem, so there is no steady state (solution oscillates forever)