ORD & PRTL DIFF EQUATIONS-MATH 353-SPRING 2015-EXAM 2

Name <u>*SOLUTIONS*</u>

Section <u>1 *Venakides*</u>

Thursday, April 16, 2015.

No Calculators. No cellphones.
Closed book and notes. Can use formula sheet.
You may use the back of the pages.
Students' "no assistance" pledge

Problem 1	20
Problem 2	20=2x10
Problem 3	20=4x5
Problem 4	20
Problem 5	20
Total	100

1. Fnd all the eigenfunction-eigenvalue pairs of the following eigenvalue problem.

$$\psi_{xx} + \lambda\psi = 0, \qquad 0 < x < \pi,$$

$$\psi_x(0) = 0, \quad \psi(\pi) = 0$$

Place your answers in a box.

CASE 1: $\lambda = 0$

General solution of ODE: $\psi(x) = c_1 x + c_2$ and thus, $\psi_x(x) = c_1$.

First BC: $\psi_x(0) = 0$, implies $c_1 = 0$. Thus, $\psi(x) = c_2$.

Second BC: $\psi(\pi) = 0$, implies $c_2 = 0$.

The solution is trivial (both c_1 and c_2 are zero) and hence inadmissible as an eigenfunction. Thus, $\lambda = 0$ is not an eigenvalue

CASE 2: $\lambda \neq 0$

General solution of ODE:

$\psi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$, hence $\psi_x(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$,

First BC: $\psi_x(0) = 0$, implies $c_2 = 0$ thus, $\psi(x) = c_1 \cos(\sqrt{\lambda}x)$

Second BC: $\psi(\pi) = 0$, implies $\cos(\sqrt{\lambda}\pi) = 0$.

Therefore

$\sqrt{\lambda_n} = n - \frac{1}{2}, \qquad n =, 1, 2, 3, \cdots$

ANSWER

$$\begin{cases} \lambda_n = (n - \frac{1}{2})^2, \qquad n =, 1, 2, 3, \cdots \\ \\ \psi_n(x) = \cos((n - \frac{1}{2})x) \end{cases} \tag{1}$$

2. You are given the PDE

$$u_{tt} + u_t - (e^{-x}u_x)_x = 0, \qquad 0 < x < \pi,$$

with homogeneous boundary conditions at $x = 0$ and $x = \pi$. You are also given that the operator $\mathbb{L}$, defined by $\mathbb{L}u = (e^{-x}u_x)_x$ with the same boundary conditions as those of the PDE is selfadjoint and that it has eigenfunctions $\psi_n(x)$ with corresponding eigenvalues λ_n, where $n = 1, 2, 3, \cdots$.

(a) If the sum

$$u(x,t) = \sum_n b_n(t)\psi_n(x)$$

satisfies the PDE and the boundary conditions, derive the ODEs that the coefficients b_n satisfy, in terms of the eigenvalues λ_n. Place your answer in a box.

The PDE can be written as

$$u_{tt} + u_t - \mathbb{L}u = 0$$

Inserting into this the above eigenfunction expansion of $u(x,t)$, using the fact that $\mathbb{L}\psi_n = \lambda_n\psi_n$ and balancing coefficients one obtains

ANSWER: $\ddot{b}_n + \dot{b}_n - \lambda_n b_n = 0 \quad$ for all n

(b) If $u(x,0) = q(x)$, where $q(x)$ is a given function, derive a fromula for $b_n(0)$ for all n.

In the given eigenfunction expansion, set $t = 0$. Take the inner product of both sides with each of the ψ_ns. Recall that the ψ_ns are mutually orthogonal because $\mathbb{L}$ is selfadjoint. Thus, $(\psi_n, \psi_m) = 0$ if $m \neq n$.

ANSWER: $\displaystyle b_n(0) = \frac{(q, \psi_n)}{(\psi_n, \psi_n)} = \frac{\int_0^\pi q(x)\psi_n(x)dx}{\int_0^\pi (\psi_n(x))^2 dx}$

3. Conisder the following initial-boundary value problem.

$$u_t = u_{xx} + e^{-t}, \qquad 0 < x < \pi, \quad t > 0$$
$$u(0,t) = 0, \qquad u(\pi,t) = 0$$
$$u(x,0) = 1$$

(a) What is the appropriate eigenfunction basis $\psi_n(x)$? What are the corresponding eigenvalues? You may skip the calculation. Place your answer in a box.

Derive by following the procedure of problem 1.

ANSWER: $\begin{cases} \psi_n(x) = \sin(nx) \\ \lambda_n = n^2 \end{cases}, \qquad n = 1, 2, 3, \cdots.$

(b) Express the nonhomogeneous term of the PDE and the initial data $u(x,0)$ as linear superpositions of the basis functions and calculate the coefficients. Place answer in box.

For both the initial data and the nonhomogeneous term of the PDE, one needs to express the constant function equal to 1 as a superposition of the eigenfunctions.

$$1 = \sum_{n=1}^{\infty} c_n \sin(nx), \qquad \text{standard calculation of } c_n \text{ gives:} \quad c_n = \begin{cases} \frac{4}{n\pi} & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases}.$$

(c) Derive an ODE for each coefficient $b_n(t)$ in the expression of the solution of the IBVP $u(x,t) = \sum_n b_n(t)\psi_n(x)$. Place your answer in a box.

PDE: $\quad u_t = -\mathbb{L}u + e^{-t}, \quad \mathbb{L}u = -u_{xx}$. Insert the series using the result of part (b):

$$\sum_{n=1}^{\infty} \dot{b}_n(t)\psi_n(x) = -\sum_{n=1}^{\infty} b_n(t)\mathbb{L}\psi_n(x) + \sum_{n=1}^{\infty} e^{-t} c_n \psi_n(x)$$

Make the replacement $\mathbb{L}\psi_n = \lambda_n \psi_n = n^2\psi_n$, then balance the coefficients of the ψ_ns.

ANSWER: $\quad \dot{b}_n = -n^2 b_n + c_n e^{-t}, \quad n = 1, 2, 3, \cdots.$

(d) Calculate the coefficients $b_n(t)$ and write down your result for the solution $u(x,t)$. Place your answer in a box.

(1) General solution of the homogeneous equation: $\qquad b_n = K_n e^{-n^2 t}$.

(2) From the result in (b) we have $b_n(0) = c_n$ for all n.

<u>When n is even</u>, $c_n = 0$. The ODE is homogeneous and $K_n = c_n = 0$. Thus, $b_n(t) = 0$.

<u>When n is odd</u>, $\qquad b_n(t) = K_n e^{-n^2 t} +$ particular solution.

Seek a particular solution $b_n = A_n e^{-t}$. One obtains $A_n = -\frac{c_n}{1-n^2}$ when $n \neq 1$. Then

$$b_n(t) = K_n e^{-n^2 t} - \tfrac{c_n e^{-t}}{1-n^2}; \quad b_n(0) = K_n - \tfrac{c_n}{1-n^2}, \quad \text{thus } b_n(0) = K_n = \tfrac{c_n}{1-n^2} + c_n$$

This obtains $b_n(t) = c_n \frac{2-n^2}{1-n^2} e^{-n^2 t} - \frac{c_n e^{-t}}{1-n^2}$ <u>when n is odd and different from 1</u>.

Finally, when $n = 1$ the guess of a scalar multiple of te^{-t} for the particular solution obtains $b_1(t) = c_1(1+t)e^{-t}$.

4. Verify that the operator $\mathbb{L}$ of taking the second derivative of functions defined on an interval (a, b), with boundary conditions $u'(a) = u(a)$ and $u'(b) = u(b)$ is selfadjoint.

Inner product: $(u, v) = \int_a^b u(x)\overline{v(x)}dx$

Must verify that all pairs of functions u and v that are defined on the interval (a, b), are twice differentiable and satisfy the above boundary conditions, satisfy the equality $(u'', v) = (u, v'')$.

We do this integrating by parts twice.

$$(u'', v) = -(u', v') + u'\overline{v}\Big|_{x=a}^{x=b} = (u, v'') - u\overline{v}'\Big|_{x=a}^{x=b} + u'\overline{v}\Big|_{x=a}^{x=b}$$

When the boundary conditions $u' = u$ and $v' = v$ at the endpoints are insertedin the last two terms, they cancel out and we obtain the desired

$$(u'', v) = (u, v'').$$

5. (a) Find the solution to the Laplace equation

$$u_{xx} + u_{yy} = 0; \quad 0 < x < \pi, \quad 0 < y < \infty.$$

You are given the following conditions :

i. $u_x = 0$ along the positive y semi-axis and along the vertical half-line $x = \pi, \ y > 0$.

ii. $u = \cos x$ on the interval $0 < x < \pi$ on the x axis.

iii. u remains bounded as y tends to infinity.

Hint: Minimal calculation is needed. Begin by sketching the region of validity of the PDE.

PDE: $\quad -\mathbb{L}u + u_{yy} = 0, \;$ where $\mathbb{L}u = -u_{xx}$

Eigenvalue problem: $\mathbb{L}\psi = \lambda\psi$ with BC at both endpoints $\psi_x = 0$.
Eigenfunctions: $1, \ \cos x, \ \cos(2x), \cos(3x), \cdots$
Only one eigenfunction, $\cos x$ participates in the condition at $y = 0$.
The solution is of the form $u = b(y)\psi(x)$. Inserting this into the PDE obtains $b_{yy} - b = 0$, which gives the general solution $b = c_1 e^{-y} + c_2 e^{y}$. We have $c_2 = 0$ to eliminate the growing solution e^y. We have $c_1 = 1$ to satisfy the condiiton at $y = 1$.

ANSWER: $\quad u(x, y) = e^{-y}\cos x.$