MATH 108/4 Fall 2009

Solutions to exam 2

(1)(a) $u''' + 8u = 0$, $u = e^{rx}$, $(r^3+8)\cancel{e^{rx}} = 0$

$r^3 + 8 = (r+2)(r^2 - 2r + 4) = 0$

$r_1 = -2$ $r_2, r_3$ : roots of $r^2 - 2r + 4 = 0$

$r_{2,3} = 1 \pm i\sqrt{3}$.

General solution : $u = c_1 e^{-2x} + \underbrace{e^{x}}_{+\infty}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x)$

as $x \to +\infty$

So $c_2 = c_3 = 0$ $u = 5e^{-2x}$, solution is unique.

(b) $u''' + 4u' = 0$ $r^3 + 4r = 0$ $r(r^2+4) = 0$

Roots: $r_1 = 0$ ; $r_{2,3} = \pm 2i$

$u(x) = c_1 + c_2 \cos 2x + c_3 \sin 2x$.

② (a)

$$\int_0^1 f(x)^2 dx = (f, f) = (a_1\phi_1 + a_2\phi_2 + \cdots, a_1\phi_1 + a_2\phi_2 + \cdots)$$

Since $\phi_1, \phi_2, \phi_3, \ldots$ is orthonormal,
the cross-terms $a_m a_n(\phi_m, \phi_n) = 0$ when $m \neq n$.
Thus,

$$(f, f) = a_1^2(\phi_1, \phi_1) + a_2^2(\phi_2, \phi_2) + \cdots = a_1^2 + a_2^2 + a_3^2 + \cdots$$

(b) $f = a_1\phi_1 + a_2\phi_2 + \cdots$
By orthogonality,
$(f, \phi_m) = a_m(\phi_m, \phi_m) = a_m$ So $(f, \phi_9) = a_9$
$(f, \phi_{15}) = a_{15}$

(c) $(f, g) = (a_1\phi_1 + a_2\phi_2 + \cdots, b_1\phi_1 + b_2\phi_2 + \cdots) =$

$$= a_1b_1\underbrace{(\phi_1, \phi_1)}_{=1} + a_1b_2\cancel{(\phi_1, \phi_2)} + a_2b_1\cancel{(\phi_2, \phi_1)} + a_2b_2\underbrace{(\phi_2, \phi_2)}_{=1} + \cdots$$

by orthogonality

$$(f, g) = a_1b_1 + a_2b_2 + \cdots$$

(3) Let $u(x,t) = b_1(t)\,\phi_1(x) + b_2(t)\,\phi_2(x) + \cdots$ (1)

where

$\phi_1(x),\ \phi_2(x)\ \ \phi_3(x)$ is an orthonormal basis that satisfies the boundary conditions

$$\phi_x(0) = 0 \quad \text{and} \quad \phi(\pi) = 0$$

The basis $\{\phi_n\}_{n=1}^{\infty}$ will be chosen along the way. Insert in the given eqn. (Notation $\frac{d}{dt} b_n(t) = \dot{b}_n$)

$$\sum_{n=1}^{\infty} \dot{b}_n \phi_n = \sum_{n=1}^{\infty} b_n \phi_n'' + \sum_{n=1}^{\infty} c_n \phi_n \qquad (2)$$

where $\sum_{n=1}^{\infty} c_n \phi_n = q(x)$

For some $m$: $\sum_{n=1}^{\infty} c_n \int_0^{\pi} \phi_n(x)\,\phi_m(x)\,dx = \int_0^{\pi} q(x)\,\phi_m(x)\,dx$

Thus: $c_m = \int_0^{\pi} q(x)\,\phi_m(x)\,dx$ (still unknown)

To have all terms of (2) expressed as series of $\phi_n$'s, we require

$$\begin{cases} \phi'' = -\lambda\phi & \lambda: \text{constant} \\ \phi_x(0) = 0 \quad \phi(\pi) = 0 \end{cases}$$

~~Solution:~~ $\cancel{\phi_n(x) = \cos\left(n - \tfrac{1}{2}\right)x \qquad n = 1, 2, 3 \ldots}$

$\cancel{\phi_n'' = -\left(n - \tfrac{1}{2}\right)^2 \cos\left(n - \tfrac{1}{2}\right)x}$

Solution $\phi_n(x) = \cos(n-\frac{1}{2})x$

$$\phi_n''(x) = -(n-\tfrac{1}{2})^2 \phi_n$$

Insert into eqn (2).

$$\sum \dot{b}_n \phi_n = -\sum b_n (n-\tfrac{1}{2})^2 \phi_n + \sum c_n \phi_n.$$

~~#~~ Balance coefficients $b_n$:

$$\dot{b}_n = -(n-\tfrac{1}{2})^2 b_n + c_n \quad \text{for } n=1,2,3 \ldots$$

Solve each ODE (for $n = 1, 2, \ldots$)

$$b_n(t) = \underbrace{\frac{c_n}{(n-\frac{1}{2})^2}}_{\text{particular solution}} + \underbrace{\left(b_n(0) - \frac{c_n}{(n-\frac{1}{2})^2}\right) e^{-(n-\frac{1}{2})^2 t}}_{\text{solves the homogenuous eqn.}}$$

$b_n(0)$ is calculated from the initial condition

$$b_1(0)\,\phi_1(x) + b_2(0)\,\phi_2(x) + \cdots = f(x)$$

~~$\delta_n$~~ To find $b_n(0)$

multiply by $\phi_n(x)$ and integrate

$$b_n(0) \underbrace{\int_0^\pi \phi_n(x)^2\,dx}_{=\frac{\pi}{2}} = \int_0^\pi f(x)\,\phi_n(x)\,dx$$

(4) (a) $u_{tt} = a^2 f''(x-at) + a^2 g''(x+at)$

$u_{xx} = f''(x-at) + g''(x+at)$

$u_{tt} - a^2 u_{xx} = 0$

(b) $u(x,0) = f(x) + g(x) = v(x)$

$u_t(x,0) = -af'(x) + ag'(x) = 0$

$f' = g'$, so $g = f + \underset{\text{constant}}{c}$

$f + g = 2f + c = v$, $f = \frac{1}{2}v - \frac{1}{2}c$

$g = \frac{1}{2}v + \frac{1}{2}c$

$u(x,t) = \frac{1}{2}v(x-at) \cancel{-\frac{1}{2}c} + \frac{1}{2}v(x+at) \cancel{+\frac{1}{2}c}$

$u(x,t) = \frac{1}{2}\left(v(x-at) + v(x+at)\right).$

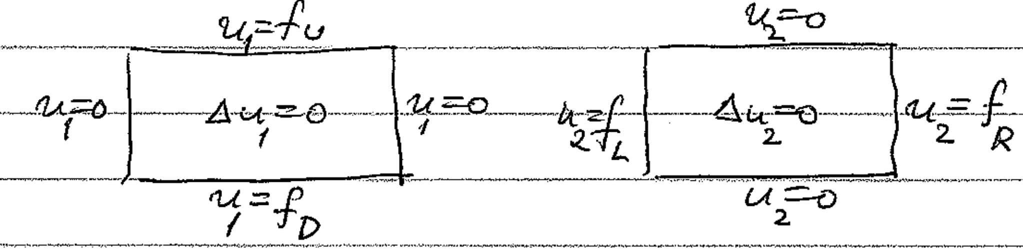
(c) $\underbrace{v(x-at)}_{u(x,t)}$ solves the wave equation.

$u(x,0) = v(x)$

$u_t(x,0) = -av'(x)$

So $h(x) = -av'(x)$

5. Solve the two problems




Solution requested: $u = u_1 + u_2$

6. Replace $x$ with $x-c$ in the first of the two given equations

$$f(x-c) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi(x-c) + b_n \sin n\pi(x-c)\right)$$

$$f(x-c) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \cos n\pi c + a_n \sin n\pi x \sin n\pi c$$

$$+ \sum_{n=1}^{\infty} b_n \sin n\pi x \cos n\pi c - b_n \cos n\pi x \sin n\pi c$$

$$f(x-c) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi c - b_n \sin n\pi c\right) \cos n\pi x$$

$$+ \sum_{n=1}^{\infty}\left(a_n \sin n\pi c + b_n \cos n\pi c\right) \sin n\pi x$$

$$A_0 = a_0$$

$$\left.\begin{aligned} A_n &= a_n \cos n\pi c - b_n \sin n\pi c \\ B_n &= a_n \sin n\pi c + b_n \cos n\pi c \end{aligned}\right\} \quad n = 1, 2, \ldots$$