Name ________________________

No Calculators.
Closed book and notes.
Write your final answers in the box if provided.
You may use the back of the pages.
Each question is worth 10 points.

MATH 108, ORD & PRTL DIFF EQUATIONS-EXAM 2

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	

1. The inner product in the set of square-integrable functions in the interval $0 < x < 1$ is defined by

$$(f, g) = \int_0^1 f(x)\overline{g(x)}dx,$$

where bar stands for the complex conjugate. Let the functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x), \cdots$ be an orthonormal basis of this set and let $f(x)$ be a known function in this set.

(a) Derive a formula (justify what you are doing, no credit for writing the formula from memory) for the coefficient a_k of the expansion

$$f(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \cdots, \qquad 0 < x < 1,$$

(b) Prove that

$$\int_0^1 |f(x)|^2 dx = |a_1|^2 + |a_2|^2 + |a_3|^2 + \cdots.$$

HINT: recall $z\overline{z} = |z|^2$.

2. Show that the sequence of functions $\cos(nx)$, where $n = 1, 2, 3, \cdots$, is NOT a basis of the set of square-integrable functions in the interval $0 < x < \pi$. HINT: Produce, as a counter-example, a function which is orthogonal to all members of the given sequence.

3. Let L be the differential operator,

$$L = -\frac{d}{dx}p(x)\frac{d}{dx}, \quad \text{thus, } Lf = -(pf')',$$

where $p(x)$ is a periodic, real valued function of period a. Let the operator act on periodic functions of period a (not necessarily real valued) that have inner product given by

$$(f,g) = \int_0^1 f(x)\overline{g(x)}dx,$$

(a) Show that if c is a scalar $(cf,g) = c(f,g)$ and $(f,cg) = \overline{c}(f,g)$.

(b) Show that $(Lf,g) = (f,Lg)$.

(c) Use the above results to show, from first principles, that if the equation

$$Lf = \lambda f,$$

holds and if f is not identically equal to zero, then, the eigenvalue λ is necessarily real. HINT: Work with the inner product (Lf,f).

4. (a) Solve the following initial boundary value problem for $u(x,t)$.

$$u_t = u_{xx} + q(x), \quad 0 < x < a, \quad t > 0,$$

$$\text{BC:} \quad u_x(0,t) = 0, \quad u(a,t) = 0,$$

$$\text{IC:} \quad u(x,0) = f(x),$$

(b) Assuming that $q(x)$ is identically equal to zero, claculate the limit,

$$\lim_{t \to +\infty} u(x,t)$$

.

5. The function $u(x, y)$, satisfies the following properties:

   (a) It satisfies Laplace's equation ($\Delta u = 0$) in the interior of a square of side a in the x,y plane, with sides parallel to the x or y coordinate axes.

   (b) It equals zero along the perimeter of the rectangle.

   Prove that $u(x, y)$ is identically zero in the square.