ORD & PRTL DIFF EQUATIONS-MATH 353-FALL 2014-EXAM 2

Name ______*KEY*______________

Section ______________________

Thursday, November 20, 2014.

No Calculators. No cellphones.
Closed book and notes. Can use formula sheet.
You may use the back of the pages.
Students' "no assistance" pledge

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Total	

1. FInd all the eigenfunction-eigenvalue pairs of the following eigenvalue problem.

$$\psi_{xx} + \lambda\psi = 0, \qquad 0 < x < \pi,$$

$$\psi(0) = 0, \quad \psi'_x(\pi) = 0$$

Place your answers in a box.

Case 1, $\lambda = 0$: $\psi = Ax + B$, $\psi(0) = B = 0$, $\psi'(\pi) = A = 0$; Thus, $\psi \equiv 0$, hence no zero e-value.

Case 2, $\lambda \neq 0$:

$$\psi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$$\psi(0) = c_1 = 0, \quad \text{Thus,} \quad \psi(x) = c_2 \sin(\sqrt{\lambda}x), \quad \psi'(x) = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x), \quad c_2 \neq 0.$$

,

$$\psi'(\pi) = \sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0, \qquad \lambda_n = (n - \tfrac{1}{2})^2; \quad n = 1, 2, 3, \cdots$$

$$\boxed{\lambda_n = (n - \tfrac{1}{2})^2, \quad \psi_n(x) = \sin(\sqrt{\lambda_n}x) = \sin((n - \tfrac{1}{2})x), \quad n = 1, 2, 3, \cdots}$$

2. (a) The function $f(x) = x$, defined on the interval $0 < x < \pi$ is represented below as a series

$$x = c_0 + \sum_{n=1}^{\infty} c_n \cos(nx).$$

Calculate the coefficients $c_0, c_1, c_2 \cdots$. Show all work. Place your answer in a box.

The function $f(x) = x$ is represented as a series of the orthogonal basis consisting of the functions, 1, $\cos(x)$, $\cos(2x)$, $\cos(3x), \cdots$ in the interval $0 < x < \pi$. In order to calculate the coefficient c_k of the expansion for any value $k = 0, 1, 2, 3, \cdots$, we multiply both sides of the expansion by the corresponding basis function and integrate over the interval, distributing these operations through the series sum.
Calculation of c_0 (basis function $\psi_0 = 1$)

$$\int_0^{\pi} x dx = c_0 \pi + \sum_{n=1}^{\infty} \int_0^{\pi} c_n \cos(nx) dx = c_0 \pi; \qquad c_0 = \tfrac{1}{\pi} \int_0^{\pi} x dx = \tfrac{\pi}{2}.$$

Calculation of c_k, $\quad k > 0$ (basis function $\psi_k = \cos(kx)$)

$$\int_0^{\pi} x \cos(kx) dx = c_0 \int_0^{\pi} \cos(kx) dx + \sum_{n=1}^{\infty} c_n \int_0^{\pi} \cos(kx)\cos(nx) dx; \quad \int_0^{\pi} x \cos(kx) dx = c_k \int_0^{\pi} \cos^2(kx) dx.$$

(all integrals, except the one with coefficient c_k, are zero by the orthogonality).
We calculate

$$\int_0^{\pi} x \cos(kx) dx = \begin{cases} 0, & \text{for even } k \\ -\frac{2}{k^2} & \text{for odd } k \end{cases}, \quad \int_0^{\pi} \cos^2(kx) dx = \tfrac{\pi}{2}, \quad \boxed{c_0 = \tfrac{\pi}{2}, \quad c_k = \begin{cases} 0, & \text{for even } k > 0 \\ -\frac{4}{\pi k^2} & \text{for odd } k. \end{cases}}$$

(b) Write down a function $u(x, y)$ that exhibits exponential growth/decay in the x direction, is oscillatory in the y direction and satisfies the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Place your answer in a box. $\boxed{u(x, y) = e^x \cos y \text{ (there are infinitely many more)}}$

(c) Find by inspection the solution of the following boundary value problem.

- $u_{xx} + u_{yy} = 0$ inside a circle in the (x, y) plane,
- At every point of the circle, the derivative of u in the direction perpendicular to the circle equals zero.

$\boxed{u(x, y) = 1}$

The solution you found is an element of the nullspace of what operator? Place your answers in a box.

$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ with Neumann boundary condition, in the context of the question}}$

(technically, any operator that annihilates the function $u = 1$)

3. Find the solution to the following initial-boundary value problem

$$u_{tt} = u_{xx} + xe^{-t}, \quad 0 < x < \pi, \quad t > 0$$
$$u_x(0,t) = 0, \quad u_x(\pi,t) = 0$$
$$u(x,0) = 1, \quad u_t(x,0) = 0$$

by the following step-by-step procedure:

(a) Imagine the solution as a (infinite) linear combination of mutually orthogonal basis functions $\psi_n(x)$ with coefficients $b_n(t)$.

$\boxed{u(x,t = \sum_n b_n(t)\psi_n(x)}$

(b) Choose the appropriate eigenfunction basis $\psi_n(x)$ (it is OK to skip the calculation). Place the eigenfuctions in a box together with the corresponding eigenvalues λ_n.

$\boxed{\psi_0 = 1,\ \psi_n = \cos(nx),\ n = 1,2,3,\cdots; \quad \lambda_0 = 0,\ \lambda_n = n^2,\ n = 1,2,3,\cdots}$

(c) Express all terms of the PDE as well as the given initial data as linear combinations of the basis functions. Place your answers in a box.

$\boxed{u_{tt} = \sum_{n=0}^{\infty} \ddot{b}_n\psi_n}$ $\boxed{u_{xx} = \sum_{n=0}^{\infty}(-\lambda_n^2 b_n\psi_n) = \sum_{n=0}^{\infty}(-n^2 b_n\psi_n)}$

$\boxed{e^{-t}x = \sum_{n=0}^{\infty} c_n e^{-t}\psi_n; \quad c_n \text{ as calculated in problem 2}}$ $\boxed{u(x,0) = 1 = \psi_0, \quad u_t(x,0) = 0}$

(d) Obtain an ODE for each coefficient b_n. Explain the reasoning behind your procedure in one or two sentences.

$\boxed{\text{We insert the above series expressions in the PDE and balance the coefficients of each basis function } \psi_n}$

(e) Derive the initial conditions for the ODE.

$$u(x,0) = 1: \quad b_0(0) = 1, \quad b_n(0) = 0, \quad n = 1,2,3,\cdots;$$
$$u_t(x,0) = 0: \quad \dot{b}_n(0) = 0, \quad n = 0,1,2,3,\cdots;$$

(f) Place the ODEs for each b_n and the corresponding initial conditions in a box.

$\boxed{\ddot{b}_0 = c_0 e^{-t}; \quad b_0(0) = 1,\ \dot{b}_0(0) = 0}$ $\boxed{\text{For } n = 1,2,3,\cdots: \quad \ddot{b}_n = -n^2 b_n + c_n e^{-t}; \quad b_n(0) = 0,\ \dot{b}_n(0) = 0}$

(g) Solve the initial value problem for the ODEs. Place the expression(s) for the coefficients $b_n(t)$ in a box.

$$\ddot{b}_0 = c_0 e^{-t}, \quad \dot{b}_0 = -c_0 e^{-t} + A_0, \quad b_0 = c_0 e^{-t} + A_0 t + B_0,$$

Using initial conditions obtains

$$\boxed{b_0 = c_0(e^{-t} + t - 1) + 1}.$$

$$\ddot{b}_n = -n^2 b_n + c_n e^{-t}, \qquad n > 0.$$

Particular solution of [NH]: $b_n = \frac{c_n}{1+n^2}e^{-t}$

General solution of [H]: $b_n = A_n\cos(nt) + B_n\sin(nt)$.

General solution of [NH]:

$$b_n = A_n\cos(nt) + B_n\sin(nt) + \tfrac{c_n}{1+n^2}e^{-t}, \quad \dot{b}_n = -nA_n\sin(nt) + nB_n\cos(nt) - \tfrac{c_n}{1+n^2}e^{-t}.$$

Implementing initial conditions: $A_n = -\frac{c_n}{1+n^2}, \quad B_n = -\frac{c_n}{n(1+n^2)}$.

$\boxed{b_n(t) = -\tfrac{c_n}{1+n^2}\cos(nt) - \tfrac{c_n}{n(1+n^2)}\sin(nt) + \tfrac{c_n}{1+n^2}e^{-t}; \qquad n = 1,2,3,\cdots}$

4. Find the solution to the Laplace equation

$$u\_\{xx\} + u\_\{yy\} = 0; \qquad 0 < x < \infty, \quad 0 < y < \pi.$$

You are given the following conditions :

(a) $u_y = 0$ along the positive x semi-axis and along the horizontal half-line $y = \pi, \ x > 0$.

(b) $u = \cos 3y$ on the interval $0 < y < \pi$ of the y axis.

(c) u remains bounded as x tends to infinity.

Hint: Can be done with little calculation. Begin by sketching the region of validity of the PDE.

Generally, we would seek a solution of the form

$$u(x, y) = \sum_n b_n(x)\psi_n(y)$$

We recognize that, with the Neumann boundary conditions at $x = 0$ and $x = \pi$, the basis functions are $1, \cos x, \cos(2x), \cdots$ The given boundary condition at $y = 0$ is exactly one of the basis functions, namely $\cos(3x)$. The coefficient for this function in the expansion will, thus, be the only nonzero coefficient. We let

$$u = b(x)\cos(3y).$$

This obviously satisfies the boundary conditons $u_y = 0$ along the positive x semi-axis and along the horizontal half-line $y = \pi, \ x > 0$.

Insert the expression for u into the PDE.

$$b_{xx}\cos(3y) - 9b\cos(3y) = 0, \qquad (b_{xx} - 9b)\cos(3x) = 0, \qquad b_{xx} - 9b = 0.$$

Solution of the ODE is

$$b = c_1 e^{-3x} + c_2 e^{3x}$$

For u to remain bounded: $c_2 = 0$, thus, $u(x, y) = c_1 e^{-3x}\cos(3y)$

Letting $x = 0$, we obtain $u(0, y) = c_1\cos(3y)$, hence $c_1 = 1$ from the given condition.

$$\boxed{u(x, y) = e^{-3x}\cos(3y)}$$

5. (a) What does it mean for an operator $\mathbb{L}$ to be self-adjoint?

$$\boxed{(\mathbb{L}u, v) = (u, \mathbb{L}v), \text{ for all } u, v \text{ in the domain of the operator } (i.e., \quad u, v \text{ satisfy the BC})}$$

(b) Verify that the second order linear differential operator $\mathbb{L} = \frac{\partial^2}{\partial x^2}$ over an interval $a < x < b$ with homogeneous boundary conditions $u(a) = 0,\ u_x(b) = 0$ is self-adjoint.

The definition of inner product and integration by parts obtain,

$$(f', g) = \int_a^b f'\bar{g}dx = -\int_a^b f\bar{g}'dx + f(b)\bar{g}(b) - f(a)\bar{g}(a) = -(f, g') + f(b)\bar{g}(b) - f(a)\bar{g}(a).$$

Thus, we have the formula

$$(f', g) = -(f, g') + f(b)\bar{g}(b) - f(a)\bar{g}(a).$$

- Using this formula with $f = u'$ and $g = v$, we obtain,

  $(u'', v) = -(u', v') + u'(b)\bar{v}(b) - u'(a)v(a) = -(u', v'), \quad$ since $u'(b) = 0$ and $v(a) = 0$

  .

- Using the formula with $f = u$ and $g = v'$, we obtain,

  $(u', v') = -(u, v'') + u(b)\bar{v}'(b) - u(a)v'(a) = -(u, v''), \quad$ since $v'(b) = 0$ and $u(a) = 0$

  .

Putting the two results together, we obtain $(u'', v) = (u, v'')$.