Math 353

Professor : Mainak Patel

Name :

MIDTERM II

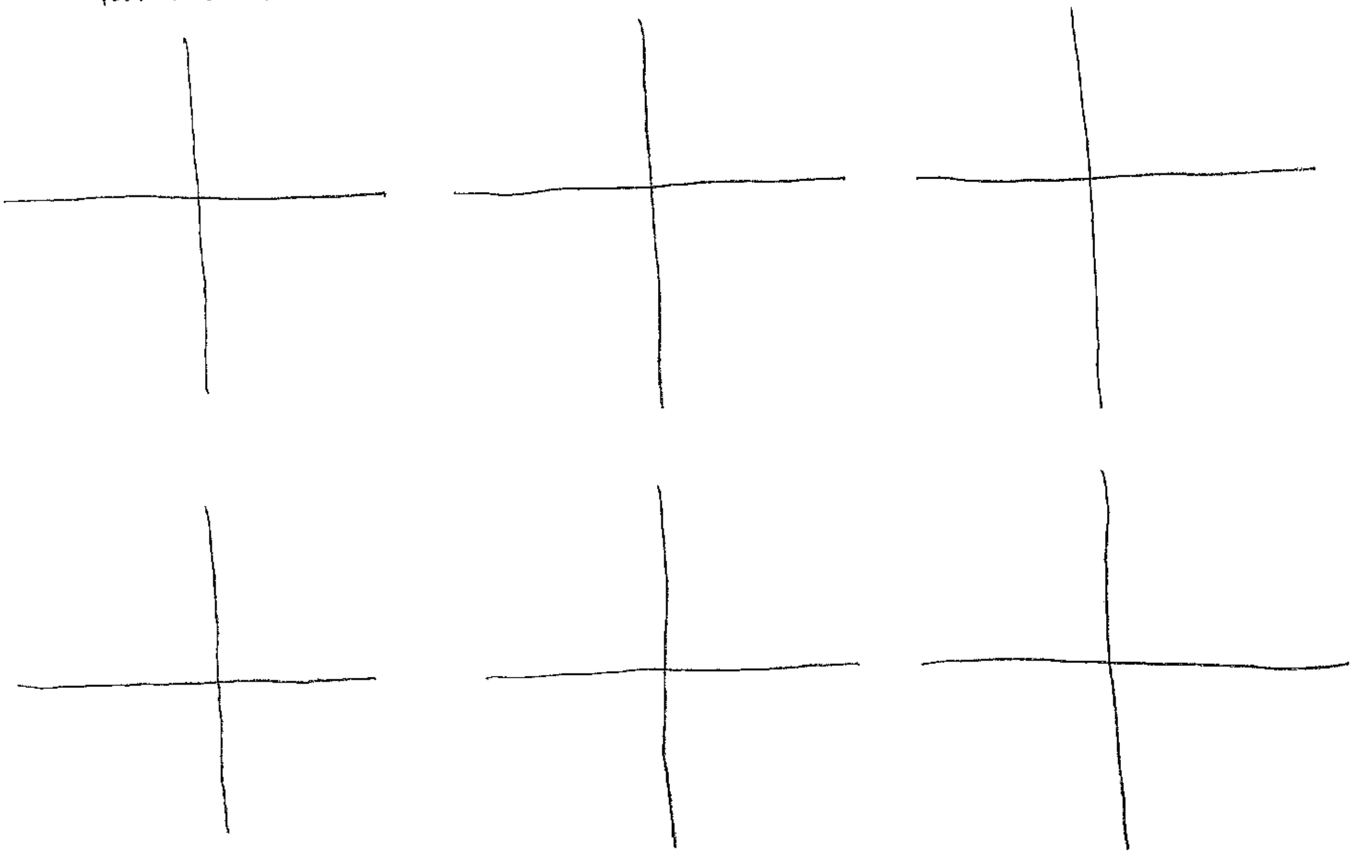
Directions : Read each question carefully and answer all parts. No calculators, notes, or devices other than your brain. <u>Show your work</u> - no credit will be given if you do not show your work. Good luck!

1. Draw the Fourier series, Fourier sine series, and Fourier cosine series of the following functions:

Fourier series | Fourier sine series | Fourier cosine series

$f(x) = |x|$ $(L=1)$

$f(x) = x$ $(L=2)$



Find a function $f(x)$ such that Fourier series of $f(x)$ = Fourier sine series of $f(x)$ = Fourier cosine series of $f(x)$.

2. Let L be a Sturm-Liouville operator. Consider the following eigenvalue problem:

$$L[y] = \lambda y \quad y'(0) = 0 \quad y'(2) = 0.$$

Let $\{\varphi_n(x)\}$ be the eigenfunctions with eigenvalues $\{\lambda_n\}$.

Define the inner product with respect to which the eigenfunctions are orthogonal:

$$\langle h_1(s), h_2(s) \rangle =$$

Suppose $\langle \varphi_n, \varphi_n \rangle = n^2$ for all n. Expand $f(x)$ in a series of eigenfunctions $\{\varphi_n\}$.

Consider the eigenvalue problem above with $L = -\frac{d^2}{dx^2}$.

Is this a Sturm-Liouville problem?

Find a set of eigenfunctions $\{\varphi_n\}$ satisfying $\langle \varphi_n, \varphi_n \rangle = n^2$ for all n.

3. Solve the following Schrödinger wave equation problem for $\Psi(x,t)$:

$$i\Psi_t = \Psi_{xx} \qquad \Psi(0,t) = 0 \qquad \Psi(\pi,t) = 0$$

(Note: $\frac{1}{i} = \frac{i}{i^2} = -i$)

$$\Psi_t(x,0) = f(x).$$

Why do we call this a wave equation rather than a heat equation?

4. Solve the following wave equation problem for $u(x,t)$:

$$u_{tt} = u_{xx} + 9\pi^2 \cos 3\pi x \qquad u_x(0,t) = 0 \quad u_x(1,t) = 0$$

$$u(x,0) = 0 \qquad u_t(x,0) = 0.$$

5. Consider the following PDE problem for $u(x,t)$:

$$u_t = x u_{xx} + u_x + t x^3 \sin \pi x$$

$$u(0,t) + 3u_x(0,t) = 0 \qquad u(x,0) = 0$$

$$u(4,t) = 0.$$

Assuming the solution can be written in the form of an eigenfunction expansion, find the eigenvalue problem for the eigenfunctions. Do not solve.

Can this problem be solved using the eigenfunction expansion method? Explain why or why not and justify your answer.

6. Consider the following nonhomogeneous wave equation problem for $u(x,t)$:

$$\left.\begin{aligned} &u_{tt} = u_{xx} + F(x,t) \\ &u(x,0) = f(x) \qquad u(0,t) = T_1 \\ &u_t(x,0) = g(x) \qquad u(L,t) = T_2. \end{aligned}\right\} \quad (*)$$

This is a problem we cannot directly solve. However, the solution $u(x,t)$ can be written as $u(x,t) = w(x,t) + v(x,t) + z(x)$, where $w(x,t)$, $v(x,t)$, and $z(x)$ are solutions to problems that we <u>can</u> solve. Write down the problems that $w(x,t)$, $v(x,t)$, and $z(x)$ solve, and show that $u(x,t) = w(x,t) + v(x,t) + z(x)$ solves $(*)$.

Can $z(x)$ be thought of as the steady state solution to $(*)$? Explain your answer.