ORD & PRTL DIFF EQUATIONS-MATH 353-Spring 2013-EXAM 2

Name KEY

Section 1, Venakides
Thursday, April 23, 2013, 10:05-11:20 AM, *Physics 119*

No Calculators.
Closed book and notes.
Place your final answers in a box.
You may use the back of the pages.

MATH 353, ORD & PRTL DIFF EQUATIONS-EXAM 1

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	

ANSWERS SHOULD BE PLACED IN A SQUARE

1. Identify the type(s) of each ODE by checking the appropriate boxes. Leave the other boxes blank (points taken off for incorrect checks). If you have identified an ODE as separable, do not check "exact" as well, in spite of the fact that "separable" implies "exact". DO NOT SOLVE THE EQUATIONS.

	linear homog.	linear nonhom.	separable	exact
$y' = xy + x + y + 1$		✓		
$(x^2 + y^3 + 2xy)dx + (3xy^2 + x^2)dy = 0$				✓
$y'^2 = y^2 - x^2$				
$y' + y\cos x = y$	✓		✓	
$y' + \cos x = y + 1$		✓		

2. (a) What does it mean to "solve the eigenvalue problem for a given square matrix A"?

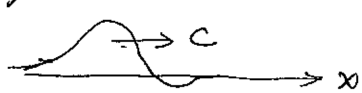
Find all pairs $(\vec{v}, \lambda)$ where $\vec{v}$ is a nonzero vector and λ is a scalar so that $A\vec{v} = \lambda\vec{v}$

(b) Write down a partial differential equation in the independent variables x and y such that $u = e^{-x}\cos y$ is one of its solutions.

$u_{xx} + u_{yy} = 0$ others too e.g. $u_x - u_{yy} = 0$

(c) Write down a function that you can characterize as a right traveling wave and state what the speed of the wave is.

$f(x-ct)$ c is the speed $c > 0$



(d) In a diffusion process, let $u(x,t)$ stand for concentration and $f(x,t)$ stand for flux. Write down the equation for the flux law.

$f = -D u_x$

EXTRA credit: Write down the mass conservation law and combine it with the flux law to derive the diffusion (heat) equation.

$u_t + f_x = 0$ (conservation of mass)

(e) Is the operator $\mathbb{L} = \frac{d^2}{dx^2} + \frac{d}{dx}$ selfadjoint in the interval $0 < x < 1$?

NO ? BC have not been specified $\Rightarrow$ Not self-adjoint?

generally check $(\mathbb{L}f, g) \stackrel{?}{=} (f, \mathbb{L}g)$ ie. $(\mathbb{L}f, g) - (f, \mathbb{L}g) = 0$

Here, it does not check even for BC $u(0) = u(1) = 0$

- works for $\frac{d^2}{dx^2}$
- fails for $\frac{d}{dx}$

3. (a) Calculate the coefficients of the expansion of the function $f(x) = x$ as a linear superposition of the basis functions $\psi_n(x) = \sin(nx)$ in the interval $0 < x < \pi$.

Basis: $\sin x, \sin(2x), \sin(3x), \sin(4x), \dots$

Representation of a function $f(x)$ $0<x<\pi$:

(*) $f(x) = c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x + \dots$

To find a formula for c_n "dot" both sides of (*) with $\sin(nx)$

$$\int_0^\pi f(x)\sin(nx)\,dx = c_1\int_0^\pi \sin x \sin(nx)\,dx + c_2\int_0^\pi \sin(2x)\sin(nx)\,dx + \dots$$

All integrals on the right are zero except $\int_0^\pi \sin^2(nx)\,dx = \frac{\pi}{2}$

so $\frac{\pi}{2} c_n = \int_0^\pi f(x)\sin(nx)\,dx$ $\boxed{c_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx}$

(b) Write down the eigenvalue problem that has the above basis as its set of eigenfunctions. Be complete in posing the problem but do not solve it.

$\mathbb{L} = -\frac{d^2}{dx^2}$

$\mathbb{L}\psi_n = \lambda_n \psi_n$ $\quad -\frac{d^2}{dx^2}\psi_n = \lambda_n \psi_n$

E-value problem:

$\psi_{xx} + \lambda\psi = 0$

BC $\psi(0) = 0$, $\psi(\pi) = 0$

4. Solve the following eigenvalue problem:

$$\psi_{xx} + \lambda\psi = 0, \qquad 0 < x < \pi.$$

$$\underbrace{\psi(0) = 0}_{BC\,1}, \quad \underbrace{\psi_x(\pi) = 0}_{BC_2}.$$

<u>Case 1:</u> $\lambda = 0$

$\psi_{xx} = 0 \qquad \psi = c_1 x + c_2 \quad , \psi_x = c_1$

<u>BC 1</u> $\quad c_2 = 0$ }
<u>BC 2</u> $\quad c_1 = 0$ } Trivial solution rejected.

<u>Case 2:</u> $\lambda \neq 0$

$\psi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \; ; \; \psi_x = \cancel{-\sqrt{\lambda}c_1 \sin\sqrt{\lambda}x} + \sqrt{\lambda}c_2 \cos\sqrt{\lambda}x$

<u>BC 1</u>: $c_1 = 0$

<u>BC 2</u>: $\sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}\,\pi) = 0$, $\cos(\sqrt{\lambda}\,\pi) = 0$ eigenvalue condition

c_2 can be anything except 0

$\sqrt{\lambda_n} = n - \frac{1}{2} \qquad n = 1, 2, 3$

$\sqrt{\lambda_n} = (n - \tfrac{1}{2})$

$\lambda_n = (n - \tfrac{1}{2})^2$

$\psi_n = c_n \sin[(n - \tfrac{1}{2})x]$

5. You are given the initial-boundary problem for the forced diffusion equation in the interval $0 < x < \pi$,

$$u_t = u_{xx} + \underbrace{\cos(x)}_{g(x) = c_0\psi_0 + c_1\psi_1 + c_2\psi_2 + \cdots} e^{-t}, \quad 0 < x < \pi,$$

with boundary conditions

$$u_x(0,t) = 0 \text{ and } u_x(\pi,t) = 0$$

and initial condition

$$u(x,0) = 1 + \cos(2x).$$

(a) Determine the solution $u(x,t)$.

(b) Calculate the limit of the solution as time tends to positive infinity.

$\frac{d^2}{dx^2} = -\mathbb{L}$

BC $\psi_x(0) = 0$, $\psi_x(\pi) = 0$

$$u_t + \mathbb{L}u = \underbrace{\cos(x)}_{c_0\psi_0 + c_1\psi_1 + c_2\psi_2 \cdots} e^{-t} \quad (*)$$

Note $c_1 = 1$ all other c_n are zero

Basis: 1, $\cos x$, $\cos(2x)$, $\cos(3x)$, ...

ψ_0, ψ_1, ψ_2, ψ_3

λ_0, λ_1, λ_2, λ_3

0, $1^2=1$, $2^2=4$, $3^2=9$

Let $u(x,t) = b_0(t)\psi_0 + b_1(t)\psi_1 + b_2(t)\psi_2 + \cdots \quad (**)$

$u(x,0) = b_0(0)\psi_0 + b_1(0)\psi_1 + b_2(0)\psi_2 + \cdots$

Note $b_0(0) = 1$, $b_2(0) = 1$ all other $b_n(0)$ are 0.

Insert $(**)$ into $(*)$ RECALL $\mathbb{L}\psi_n = \lambda_n\psi_n$

$$\boxed{\begin{aligned}&\sum_{n=0}^{\infty}(\dot{b}_n + \lambda_n b_n)\psi_n = \sum_{n=0}^{\infty}(c_n e^{-t})\psi_n \qquad n = 0, 1, 2, 3 \ldots \\ &\dot{b}_n + \lambda_n b_n = c_n e^{-t} \qquad n = 0, 1, 2, 3 \ldots\end{aligned}}$$

$n = 3, 4, 5, \ldots$ $\quad \dot{b}_n + n^2 b_n = 0 \quad b_n(0) = 0$, $\quad b_n(t) = b_n(0)e^{-n^2t} = 0$

$b_n(t) = 0 \quad n = 3, 4, \ldots$

$n = 0$ $\quad \dot{b}_0 = 0 \quad b_0(t) = b_0(0) = 1 \quad b_0(t) = 1$

$n = 1$ $\quad \dot{b}_1 + b_1 = e^{-t} \quad b_1 = \underbrace{b_1(0)}_{=0}e^{-t} + te^{-t} \Rightarrow b_1(t) = te^{-t}$

$n = 2$ $\quad \dot{b}_2 + 4b_2 = 0 \quad b_2(t) = b_2(0)e^{-4t} \quad b_2(t) = e^{-4t}$

$(u(x,t) = 1 + te^{-t}\cos x + e^{-4t}\cos 2x)$

(b) $u(x,t) \to 1$ as $t \to \infty$

6. (a) Verify that the function $u(x,t) = f(x-at) + g(x+at)$, where f and g are arbitrary functions, satisfies the wave equation $u_{tt} - a^2 u_{xx} = 0$.

$u_{xx} = f''(x-at) + g''(x+at)$

$u_{tt} = a^2[f''(x-at) + g''(x+at)]$

clearly $u_{tt} - a^2 u_{xx} = 0$

(b) Find the solution of the initial value problem of the wave equation on the full line $-\infty < x < \infty$

$$u_{tt} - a^2 u_{xx} = 0, \quad -\infty < x < \infty,$$

with initial conditions

$$u(x,0) = \frac{1}{x^2+1}, \quad u_t(x,0) = 0.$$

$u(x,t) = f(x-at) + g(x+at)$

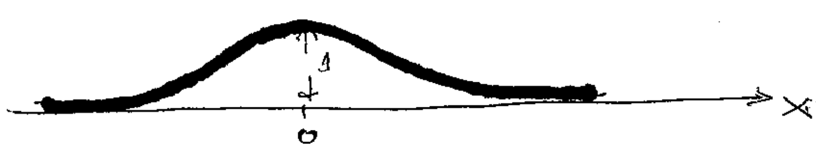
$u_t(x,t) = -a f'(x-at) + a g'(x+at)$

$u(x,0) = f(x) + g(x) = \frac{1}{x^2+1}$

$u_t(x,0) = -a f'(x) + a g'(x) = 0 \implies f(x) = g(x) + c$

(c) Draw the graph of the initial data

$u(x,t) = \frac{1}{2}\left[\frac{1}{(x-at)^2+1} + \frac{1}{(x+at)^2+1}\right]$



(d) Graph of the solution at a later time.