Name KEY

No Calculators.
Closed book and notes.
Write your final answers in the box if provided.
You may use the back of the pages.

MATH 108, ORD & PRTL DIFF EQUATIONS-EXAM 2

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1. The inner product in the vector space of real functions in the interval $0 < x < 1$ is defined by

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Let the functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x), \cdots$ be an orthogonal (not necessarily orthonormal) basis of this space and let $f(x)$ be a known function in this set.

(a) Derive a formula for the coefficient a_n of the expansion

$$f(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \cdots, \qquad 0 < x < 1,$$

(no credit for writing the formula from memory)

$$(f, \phi_n) = a_1(\phi_1, \phi_n) + \cdots + a_n(\phi_n, \phi_n) + \cdots$$

$$(\phi_m, \phi_n) = 0 \quad \text{when } m \neq n$$

$$(f, \phi_n) = a_n(\phi_n, \phi_n) \quad , \quad a_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}$$

$$a_n = \frac{\int_0^1 f(x)\overline{\phi_n(x)}\,dx}{\int_0^1 |\phi_n(x)|^2 dx}$$

(b) Given a second real function, and now assuming that the basis is ORTHONORMAL,

$$g(x) = b_1\phi_1(x) + b_2\phi_2(x) + b_3\phi_3(x) + \cdots, \qquad 0 < x < 1,$$

prove that

$$(f, g) = a_1b_1 + a_2b_2 + a_3b_3 + \cdots.$$

$$(f, g) = (a_1\phi_1 + a_2\phi_2 + \cdots, b_1\phi_1 + b_2\phi_2 + \cdots)$$

$$= a_1\overline{b_1} + a_2\overline{b_2} + \cdots$$

$$\text{since } (\phi_m, \phi_n) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

2. (a) Calculate the Fourier series of the function $f(x)$ that equals x^2 when $-\pi \le x \le \pi$ and is repeated periodically with period 2π over the x axis.

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \underbrace{\sum_{n=1}^{\infty} b_n \sin(nx)}_{=0 \text{ since } x^2 \text{ is even}}$$

$$n=0 \qquad \int_{-\pi}^{\pi} x^2 dx = \frac{a_0}{2} \cdot 2\pi \qquad a_0 = \frac{2\pi^2}{3}$$

$$n \geq 1 \qquad a_n \underbrace{\int_{-\pi}^{\pi} \cos^2(nx) dx}_{=\pi} = \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx = \frac{1}{n} \int_{-\pi}^{\pi} x^2 (\sin(nx))' dx$$

$$= -\frac{2}{n} \int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{2}{n^2} \int_{-\pi}^{\pi} x (\cos(nx))' dx$$

$$= \frac{2}{n^2} x \cos(nx) \Big|_{-\pi}^{\pi} - \frac{2}{n^2} \underbrace{\int_{-\pi}^{\pi} \cos(nx) dx}_{=0} = \frac{4\pi}{n^2}(-1)^n$$

So: $a_n = \frac{4}{n^2}(-1)^n \quad n = 1, 2, \dots$

$$x^2 = \frac{\pi^2}{3} + 4 \underbrace{\left[-\cos x + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \dots \right]}_{\text{Series } S(x)}$$

(b) Calculate the series

$$A = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$$

and the series

$$B = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

Series $A = -S(x)\big|_{x=0}$, Series $B = S(x)\big|_{x=\pi}$

So $x=0 \qquad \frac{\pi^2}{3} - 4A = 0 \qquad A = \frac{\pi^2}{12}$

$x=\pi \qquad \frac{\pi^2}{3} + 4B = \pi^2 \qquad B = \frac{\pi^2}{6}$

3. (a) Solve the eigenvalue problem (find all eigenvalues and the corresponding eigenfunctions)

$$\phi'' + \lambda\phi = 0, \quad 0 \le x \le 1,$$

with boundary conditions $\phi(0) = 0$ and $\phi'(1) = 0$. NOTE: When determining the eigenvalues, examine all three cases, $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$

$\lambda = 0 \qquad \phi = Ax + B \qquad \left.\begin{array}{l}\phi(0) = 0 \Rightarrow B = 0 \\ \phi'(1) = 0 \Rightarrow A = 0\end{array}\right\}$ $\lambda = 0$ is not an eigenvalue

$\lambda > 0 \quad \phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \,; \; \phi' = -\sqrt{\lambda}c_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x)$

$\phi(0) = 0 \Rightarrow c_1 = 0 \;;\; \phi'(1) = 0 \Rightarrow \cos\sqrt{\lambda} = 0$

$\sqrt{\lambda_n} = \pi n - \frac{\pi}{2}, \; \phi_n = \sin((n - \frac{1}{2})\pi x), \; n = 1, 2, \ldots$

$\lambda < 0 \quad \phi = c_1 \cosh(\sqrt{|\lambda|}x) + c_2 \sinh(\sqrt{|\lambda|}x); \; \phi'(x) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}x) + c_2\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x)$

$\phi(0) = 0 \Rightarrow c_1 = 0 \;;\; \phi'(1) = 0 \Rightarrow c_2 = 0$

(since $\cosh(\sqrt{\lambda}x) \neq 0 \;\; \forall x$.

thus, no eigenvalues $\lambda < 0$

Ans $\left.\begin{array}{l}\lambda_n = (n - \frac{1}{2})^2\pi^2 \\ \phi_n(x) = \sin((n - \frac{1}{2})\pi x)\end{array}\right\} n = 1, 2, 3 \ldots$

(b) Let L be the differential operator,

$$L = \frac{d^2}{dx^2} + \frac{d}{dx},$$

with periodic boundary conditions over the interval $[0, a]$. Is the operator selfadjoint? Justify your answer.

$$L = \underbrace{\frac{d^2}{dx^2}}_{L_1} + \underbrace{\frac{d}{dx}}_{L_2}$$

L_1 (i.e. $\frac{d^2}{dx^2}$) is selfadjoint, it has form $\frac{d}{dx}p(x)\frac{d}{dx} + q(x)$ with $p(x) = 1 \quad q(x) = 0$

L_2 is NOT $\quad (L_2 f, g) = \int_0^a f'\bar{g}\,dx = -\int_0^a f\bar{g}'\,dx = -\int_0^a f\bar{g}' = -(f, L_2 g)$

There are no boundary terms due to periodic B.C.

4. Solve the following initial boundary value problem for $u(x,t)$.

$$u_t = u_{xx} + q(x)e^{-t}, \quad 0 < x < a, \quad t > 0,$$

$$\text{BC:} \quad u_x(0,t) = 0, \quad u_x(a,t) = 0,$$

$$\text{IC:} \quad u(x,0) = x - \frac{a}{2},$$

NOTE: When determining the eigenvalues, examine all three cases, $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$. use orthogonal basis $\phi_1, \phi_2, \dots$

$u(x,t) = b_1(t)\phi_1(x) + b_2(t)\phi_2(x) + \cdots$; $\frac{d^2}{dx^2} = \mathbb{L}$; $q(x) = c_1\phi_1 + c_2\phi_2 + \cdots$

PDE: $\dot{b}_1\phi_1 + \dot{b}_2\phi_2 + \cdots = b_1\mathbb{L}\phi_1 + b_2\mathbb{L}\phi_2 + \cdots + e^{-t}(c_1\phi_1 + c_2\phi_2 + \cdots) \quad (*)$

Selection of basis : $\mathbb{L}\phi = -\lambda\phi$ (that is $\phi_{xx} + \lambda\phi = 0$)

$$\phi_x(0) = 0, \quad \phi_x(a) = 0$$

$\lambda = 0$: e-value $\lambda_0 = 0$, e-function $\phi_0 = 1$; $\lambda > 0$ e-values $\lambda_n = \left(\frac{n\pi}{a}\right)^2$, e-functions $\phi_n = \cos\frac{n\pi x}{a}$; $n = 1, 2, \dots$

Insert $\mathbb{L}\phi_n = -\lambda\phi_n$ into PDE (Eq. $*$), and collect terms

$$\sum_{n=0}^{\infty} \underbrace{(\dot{b}_n + \lambda_n b_n - c_n e^{-t})}_{=0 \text{ since } \phi_0, \phi_1, \phi_2, \dots \text{ is a basis}} \phi_n = 0$$

ODEs

$\dot{b}_n + \lambda_n b_n = c_n e^{-t} \qquad n = 0, 1, 2, \dots$

Integr. factor : $e^{\lambda_n t}$

$(b_n e^{\lambda_n t})' = c_n e^{(\lambda_n - 1)t}$

$b_n e^{\lambda_n t} = \frac{c_n e^{(\lambda_n - 1)t}}{\lambda_n - 1} + k_n$ if $\lambda_n \neq 1$

$$\begin{cases} b_n(t) = \frac{c_n e^{-t}}{\lambda_n - 1} + k_n e^{-\lambda_n t} & \text{if } \lambda_n \neq 1 \\ b_n(t) = c_n t e^{-t} + k_n e^{-t} & \text{if } \lambda_n = 1 \end{cases}$$

$c_n = \frac{(q, \phi_n)}{(\phi_n, \phi_n)}$

$b_n(0) = \frac{(x - \frac{a}{2}, \phi_n)}{(\phi_n, \phi_n)}$

Find k_n by letting $t = 0$ in equations

5. You are given the following initial boundary value problem for $u(x,t)$ (same as previous problem).

$$u_t = u_{xx} + q(x)e^{-t}, \quad 0 < x < a, \quad t > 0, \qquad (*)$$

$$\text{BC:} \quad u_x(0,t) = 0, \quad u_x(a,t) = 0,$$

$$\text{IC:} \quad u(x,0) = x - \frac{a}{2},$$

(a) Assuming that $\lim_{t\to+\infty} u(x,t)$ exists, show that it is constant (independent of x), without using the series expansion.

Taking the limit $t \to \infty$ in the PDE above (Eq. $*$), and assuming $u_t \to 0$, we obtain in the limit:

$u_{xx} = 0$, $u_\infty = Ax + B$. Bound. Cond. $\Rightarrow A = 0$

Thus, $u_\infty = B = \text{constant}$.

Note that the implication $u(t)$ converges $\Rightarrow u_t \to 0$ is not generally true (example $u(t) = \frac{\sin t^2}{t}$ as $t \to \infty$.

(b) Derive an ODE for the quantity $V(t)$ given by

$$V(t) = \int_0^a u(x,t), \qquad \left(\text{assume,} \quad \int_0^a q(x)dx = Q\right),$$

and solve its initial value problem.

Integrate the PDE with respect to x over the interval and interchange the order of the t-differentiation and integration (not always permissible).

$$\frac{d}{dt}\int_0^a u\,dx = \underbrace{u_x\Big|_0^a}_{=0 \text{ (by BC)}} + e^{-t}\int_0^a q(x)\,dx; \quad \frac{dV}{dt} = e^{-t}Q; \quad V(0) = \underbrace{\int_0^a \left(x - \frac{a}{2}\right)dx}_{=0}$$

$$V(t) = Q(1 - e^{-t})$$

(c) Use your results to calculate the limit $\lim_{t\to+\infty} u(x,t)$.

$$V(\infty) = Q$$

$$V(\infty) = \int_0^a B\,dx = \frac{Ba}{2} \qquad \text{So } \frac{Ba}{2} = Q, \quad B = \frac{2Q}{a}.$$

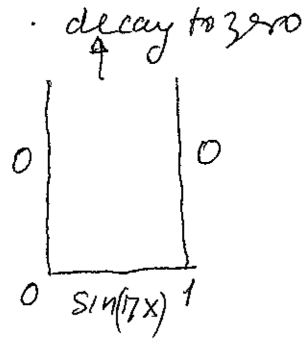
$$u_\infty = \frac{2Q}{a}$$

6. Solve Laplace's equation (CAN BE DONE WITHOUT EXPANDING TO A SERIES)

$$u_{xx} + u_{yy} = 0$$

in the region $0 \leq x \leq 1$ and $y \geq 0$, with the following boundary conditions:

$u = 0$, when $x = 0$ and when $x = 1$; $u(x, 0) = sin(\pi x)$; u converges to zero, when y tends to infinity.



$\sin(\pi x)$ is an eigen function for the given BC.

$u = b(y)\sin(\pi x)$ insert into PDE

$$-\pi^2 b \sin(\pi x) + b_{yy} \sin(\pi x) = 0$$

$$b_{yy} - \pi^2 b = 0 \qquad b = c_1 e^{-\pi y} + c_2 e^{\pi y}$$

$c_2 = 0$ since $e^{\pi y} \to \infty$ as $y \to \infty$

Ans: $u(x,y) = e^{-\pi y} \sin \pi x$