Name ________________________

No Calculators.
Closed book and notes.
Write your final answers in the box if provided.
You may use the back of the pages.

MATH 108, ORD & PRTL DIFF EQUATIONS-EXAM 2

Problem 1	
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1. The inner product in the vector space of real functions in the interval $0 < x < 1$ is defined by

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Let the functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x), \cdots$ be an orthogonal (not necessarily orthonormal) basis of this space and let $f(x)$ be a known function in this set.

(a) Derive a formula for the coefficient a_n of the expansion

$$f(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \cdots, \qquad 0 < x < 1,$$

(no credit for writing the formula from memory)

(b) Given a second real function, and now assuming that the basis is ORTHONORMAL,

$$g(x) = b_1\phi_1(x) + b_2\phi_2(x) + b_3\phi_3(x) + \cdots, \qquad 0 < x < 1,$$

prove that

$$(f, g) = a_1b_1 + a_2b_2 + a_3b_3 + \cdots.$$

2. (a) Calculate the Fourier series of the function $f(x)$ that equals x^2 when $-\pi \leq x \leq \pi$ and is repeated periodically with period 2π over the x axis.

(b) Calculate the series

$$A = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$$

and the series

$$B = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

3. (a) Solve the eigenvalue problem (find all eigenvalues and the corresponding eigenfunctions)

$$\phi'' + \lambda\phi = 0, \quad 0 \leq x \leq 1,$$

with boundary conditions $\phi(0) = 0$ and $\phi'(1) = 0$. NOTE: When determining the eigenvalues, examine all three cases, $\lambda = 0, \;\; \lambda > 0$ and $\lambda < 0$

(b) Let L be the differential operator,

$$L = \frac{d^2}{dx^2} + \frac{d}{dx},$$

with periodic boundary conditions over the interval $[0, a]$. Is the operator selfadjoint? Justify your answer.

4. Solve the following initial boundary value problem for $u(x,t)$.

$$u_t = u_{xx} + q(x)e^{-t}, \quad 0 < x < a, \quad t > 0,$$

$$\text{BC:} \quad u_x(0,t) = 0, \quad u_x(a,t) = 0,$$

$$\text{IC:} \quad u(x,0) = x - \frac{a}{2},$$

NOTE: When determining the eigenvalues, examine all three cases, $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$.

5. You are given the following initial boundary value problem for $u(x,t)$ (same as previous problem).

$$u_t = u_{xx} + q(x)e^{-t}, \quad 0 < x < a, \quad t > 0,$$

$$\text{BC:} \quad u_x(0,t) = 0, \quad u_x(a,t) = 0,$$

$$\text{IC:} \quad u(x,0) = x - \frac{a}{2},$$

(a) Assuming that $\lim_{t\to+\infty} u(x,t)$ exists, show that it is constant (independent of x), without using the series expansion.

(b) Derive an ODE for the quantity $V(t)$ given by

$$V(t) = \int_0^a u(x,t), \qquad \left(\text{assume,} \quad \int_0^a q(x)dx = Q\right),$$

and solve its initial value problem.

(c) Use your results to calculate the limit $\lim_{t\to+\infty} u(x,t)$.

6. Solve Laplace's equation (CAN BE DONE WITHOUT EXPANDING TO A SERIES)

$$u_{xx} + u_{yy} = 0$$

in the region $0 \leq x \leq 1$ and $y \geq 0$, with the following boundary conditions:

$u = 0$, when $x = 0$ and when $x = 1$; $u(x, 0) = sin(\pi x)$; u converges to zero, when y tends to infinity.

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