/

Asst. Professor Storm December 11, 2006

Math 53

Final Exam

Instructions. Answer the following problems carefully and completely. Make sure you show all your work. Do not use a calculator. In problems 3, 7, and 8 you may use the FACT without justification. There are 100 points possible. Good luck!

Name SOLUTIONS

TA's name and time ____________________

1.	(10)	______
2.	(12)	______
3.	(10)	______
4.	(13)	______
5.	(12)	______
6.	(9)	______
7.	(10)	______
8.	(15)	______
9.	(9)	______
Total	(100)	______

1. Suppose $y(t)$ is a function satisfying the equations

$$y'(t) = e^{y(t)+t} \quad \text{and} \quad y(0) = 0.$$

Find an explicit formula for $y(t)$.

Divide both sides by e^y

$\rightsquigarrow y'e^{-y} = e^t$

$\rightsquigarrow \int y'e^{-y}\,dt = \int e^t\,dt$

$\rightsquigarrow -e^{-y} = e^t + C$

$y(0) = 0 \Rightarrow -1 = 1 + C$ & $C = -2$

$\therefore e^{-y} = 2 - e^t$

If $t < \ln 2$ can take ln to get

$y = -\ln(2 - e^t) \quad (t < \ln 2)$

2. Consider the second order ODE

$$y''(t) + 4y(t) = -2\cos(2t).$$

Without using Laplace transforms, find a solution to this ODE.

$\boxed{1}$ Use method of undetermined coeff's.

Try: $y(t) = A\cos 2t + B\sin 2t$

$\rightarrow y'(t) = -2A\sin 2t + 2B\cos 2t$

$y''(t) = -4A\cos 2t - 4B\sin 2t$

$\therefore y'' + 4y = -4A\cos 2t - 4B\sin 2t$
$+ 4(A\cos 2t + B\sin 2t)$
$= 0 \neq -2\cos 2t$

This fails, now try

$y(t) = At\cos 2t + Bt\sin 2t$

$\rightarrow y'(t) = A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t$

$\rightarrow y''(t) = -2A\sin 2t - 2A\sin 2t - 4At\cos 2t$
$+ 2B\cos 2t + 2B\cos 2t - 4Bt\cos 2t$

$\therefore y'' + 4y = -4A\sin 2t + 4B\cos 2t = -2\cos 2t$

$\therefore A = 0,\ B = -\frac{1}{2}$ & so $y = -\frac{t}{2}\sin 2t$ is a solution.

[2] Use variation of parameters (longer, more involved!)

Easy to solution to homogeneous problem is:

$y_h(t) = A\cos 2t + B\sin 2t \Rightarrow$ try $y_p(t) = v_1\cos 2t + v_2 \sin 2t$

Now, $y_p'(t) = v_1'\cos 2t - 2v_1 \sin 2t + v_2'\sin 2t + 2v_2\cos 2t$

Scratch paper

1st condition: $v_1'\cos 2t + v_2'\sin 2t = 0$

$\Rightarrow y_p''(t) = -2v_1'\sin 2t - 4v_1\cos 2t + 2v_2'\cos 2t - 4v_2\sin 2t$

$\therefore y_p'' + 4y_p = -2\cos 2t \Rightarrow$ 2nd condition: $v_1'\sin 2t - v_2'\cos 2t = \cos 2t$

Hence, have the system:

$$\begin{bmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix} \Rightarrow \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{1}{-\cos^2 2t - \sin^2 2t}\begin{bmatrix} -\cos 2t & -\sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}\begin{bmatrix} 0 \\ \cos \end{bmatrix}$$

$\therefore v_1' = \sin 2t\cos 2t = \frac{1}{2}\sin 4t$ (RECALL: $\sin 2w = 2\sin w\cos w$)

$\rightsquigarrow v_1 = -\frac{1}{8}\cos 4t$

& $v_2' = -\cos^2 2t = -\frac{1}{2}[\cos 4t + 1]$ (RECALL: $\cos 2w = 2\cos^2 w - 1$)

$\rightsquigarrow v_2 = -\frac{1}{8}\sin 4t - \frac{t}{2}$

$\therefore y_p(t) = \left(-\frac{1}{8}\cos 4t\right)\cos 2t + \left(-\frac{1}{8}\sin 4t - \frac{t}{2}\right)\sin 2t$

$= -\frac{t}{2}\sin 2t - \frac{1}{8}\left(\cos 4t\cos 2t + \sin 4t\sin 2t\right)$

But, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ & so taking $\alpha = 4t$, $\beta = 2t$ we see

$y_p(t) = -\frac{t}{2}\sin 2t - \frac{1}{8}\cos 2t$

This last term is a homogeneous solution & consequently we may take

$y_p(t) = -\frac{t}{2}\sin 2t$

3. Consider the system $\vec{y}\,'(t) = A\vec{y}(t)$ where

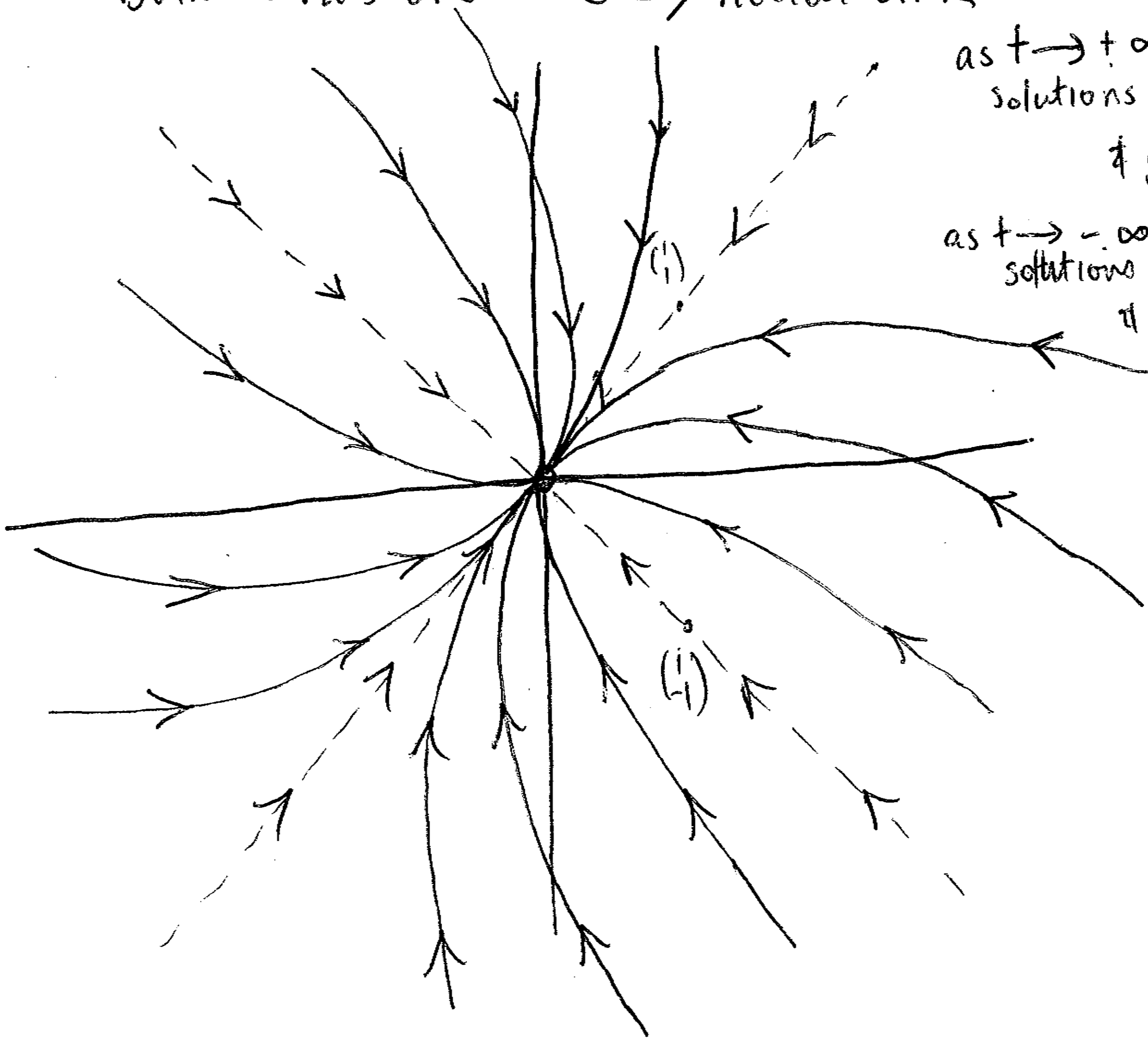
$$A = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix}.$$

FACT: This system has the following fundamental set of solutions:

$$\left\{ e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \ e^{-6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Classify the equilibrium at the origin and sketch the phase-plane portrait for the above system. In particular draw a few arrows on the solution curves to illustrate the direction of the flow and explain the behavior as $t \to \infty$ and $t \to -\infty$.

Both e-vals are -ve $\Rightarrow$ nodal sink



as $t \to +\infty$
solutions $y \to 0$
& $y' \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

as $t \to -\infty$
solutions y are unbounded
& y' becomes parallel to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. Consider the second order initial value problem

$$y''(t) - y(t) = 2t, \quad y(0) = 0, \quad y'(0) = 1.$$

Use the Laplace transform to find the solution of this initial value problem.

$$\mathcal{L}\{y''-y\} = \mathcal{L}\{2t\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) - Y(s) = \frac{2}{s^2}$$

$$(s^2-1)Y(s) - 1 = \frac{2}{s^2}$$

$$Y(s) = \frac{s^2+2}{s^2(s^2-1)} = \boxed{\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}}$$

$$\circledast \quad s^2+2 = s(s^2-1)A + (s^2-1)B + s^2(s+1)C + s^2(s-1)D$$

$\circledast|_{s=0}$: $\quad 2 = -B \rightsquigarrow B = -2$

$\circledast|_{s=1}$: $\quad 3 = 2C \rightsquigarrow C = \frac{3}{2}$

$\circledast|_{s=-1}$: $\quad 3 = -2D \rightsquigarrow D = -\frac{3}{2}$

$\circledast'$: $\quad 2s = (3s^2-1)A + 2sB + (3s^2+2s)C + (3s^2-2s)D$

$\circledast'|_{s=0}$: $\quad 0 = -A \rightsquigarrow A = 0$

so $\quad Y(s) = -\frac{2}{s^2} + \frac{3/2}{s-1} - \frac{3/2}{s+1}$

and $\quad \boxed{y(t) = -2t + \frac{3}{2}e^{t} - \frac{3}{2}e^{-t}}$

$\quad = -2t + 3\cosh t.$

5. Consider the differential equation

$$\circledast \quad \underbrace{\sin(2y)}_{P}dx + \underbrace{(2x\cos(2y) - 2y)}_{Q}dy = 0.$$

Determine if this equation is exact. If it is exact then solve it, i.e. find a function $F(x, y)$ so that the solutions of the ODE are the level sets of $F(x, y)$.

Check: $\quad \dfrac{\partial P}{\partial y} \stackrel{?}{=} \dfrac{\partial Q}{\partial x}$

$$2\cos 2y \stackrel{\checkmark}{=} 2\cos 2y$$

$\circledast$ is exact. Need F where $\overset{1)}{\dfrac{\partial F}{\partial x}} = \sin 2y$, $\overset{2)}{\dfrac{\partial F}{\partial y}} = 2x\cos 2y - 2y$.

From 2), $F(x,y) = \int (2x\cos 2y - 2y)\, dy = x \sin 2y - y^2 + \psi(x)$.

$\dfrac{\partial F}{\partial x} = \sin 2y + \psi'(x)$, where $P = \sin 2y$; so take $\psi \equiv 0$.

We have $F = \boxed{x \sin 2y - y^2 \boxed{= C}}$.

6. Define a system of first order ODEs equivalent to the single ODE

$$x'''(t) + x''(t)x(t) - x'(t)x(t) = 2.$$

It is not necessary to solve the resulting system.

order is 3, so need 3 dimensions.

$$x = x \qquad \begin{cases} x' = y \\ y' = z \\ z' = 2 + xy - xz \end{cases}$$
$$y = x' \quad \rightsquigarrow$$
$$z = x''$$

$$z' = x''' = 2 + x'x - x''x$$
$$= 2 + xy - xz$$

Solution to 7.

(a)

$$\begin{pmatrix} x_1(t) \\ y_1(t) \\ z_1(t) \end{pmatrix} = \begin{pmatrix} b\cos t \\ 0 \\ b\sin t \end{pmatrix}$$

(b)

$$\begin{pmatrix} x_2(0) \\ y_2(0) \\ z_2(0) \end{pmatrix} = \begin{pmatrix} 14 \\ 42 \\ -4 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow C_1 = 42 \Rightarrow y_2(t) = C_1 e^{-t} = 42e^{-t}.$$

$e^{-t} > 0$ $\cancel{\forall}$ for all t.

So $42e^{-t} > 0$ for all t.

YES.

8. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 3 & -2 & 3 \end{pmatrix}.$$

FACTS: The characteristic equation for A is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0.$$

2 is an eigenvalue of A. Finally,

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}.$$

Find a fundamental set of solutions to the first order linear system

$$\vec{y}\,'(t) = A\vec{y}(t).$$

First I must find the eigenvalues. I know 2 is an e-value.

$$\begin{array}{r} \lambda^2 - 4\lambda + 4 \\ \lambda - 2 \,\big)\, \overline{\lambda^3 - 6\lambda^2 + 12\lambda - 8} \\ \underline{\lambda^3 - 2\lambda^2} \\ -4\lambda^2 \\ \underline{-4\lambda^2 + 8\lambda} \\ 4\lambda \\ \underline{4\lambda - 8} \\ 0 \end{array}$$

So the char. eqn. is

$$(\lambda-2)(\lambda^2 - 4\lambda + 4) = (\lambda-2)^3 = 0.$$

Next I'll find an e-vector for 2.

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & -2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

So $e^{2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is a sol'n.

Next I use the FACT

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & -2 & 1 \end{pmatrix}^2 = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 1 & -1 \end{pmatrix} \rightsquigarrow -2y_1 + y_2 - y_3 =$$

I choose $y_1 = 0$. Thus $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a generalized e-vector for A.

$$e^{tA}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t(A-2I)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right] = e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right]$$ is a sol'n.

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & -2 & 1 \end{pmatrix}^3 = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So any vector is a generalized e-vector. I choose $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$e^{tA}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{2t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}\right].$$

$$= e^{2t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + t^2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right]$$ is a sol'n.

To ensure I've found a fundamental set of solutions, let me check that my three vectors form a basis.

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = 1\cdot(-1-1) = -2 \neq 0.$$

Therefore a fundamental set of solutions is

$$e^{2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right], \quad \text{and}$$

$$e^{2t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + t^2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right].$$

(Note: There are significantly shorter correct solutions to this problem.)

9.

Step #	t	y'	Euler output
0	0	0	0
1	0.5	0.5	$0 + 0 \times 0.5 = 0$
2	1	1.0	$0 + 0.5 \times 0.5 = 0.25$
3	1.5	1.5	$0.25 + 1.0 * 0.5 = 0.75$ multiplication
4	2	2.0	$0.75 + 1.5 \times 0.5 = 1.5$

Using Euler's method I obtain the estimate $y(2) \approx 1.5$.