Stanford University Department of Mathematics

Math 53, Fall 2007 — Final Exam

Instructor : Samuel Lisi

Date: 10 December 2007
Duration: 3 hours

FAMILY NAME : ______________________________

GIVEN NAME(S) : ______________________________

STUDENT NUMBER : ______________________________

YOUR SIGNATURE: ______________________________

DO NOT OPEN THIS TEST UNTIL INSTRUCTED TO DO SO.

INSTRUCTIONS:

- Your signature above indicates that you have abided by the Stanford Honor Code while writing this test.
- There are eight questions. The last part of question 8 is a bonus.
- You may quote theorems from your textbook or from class if you make an appropriate reference.
- Show all your work.
- No electronic devices of any kind (e.g. calculators, cell-phones) are allowed.
- There is a table of Laplace transform identities on the last page of the exam. You may use these identities without proof, unless the question indicates otherwise.

Question	Marks
1 [10 pts]	
2 [10 pts]	
3 [15 pts]	
4 [15 pts]	
5 [15 pts]	
6 [15 pts]	
7 [15 pts]	
8 [20 pts]	
Total [100 points]	

1. (a) i. Find a solution to the initial value problem [10 pts]

$$y' = ty^2, \quad y(0) = 1.$$

ii. What is the interval of existence of this solution?

iii. Is this solution unique? If yes, explain why. If no, provide a second solution.

(b) Find the general solution to the differential equation

$$y' + 2ty = e^{-t^2}\sin(t).$$

(a) (i.) Separate variables.

$$\frac{y'}{y^2} = t. \qquad \int \frac{dy}{y^2} = \int t\,dt \qquad -\frac{1}{y} = \frac{1}{2}t^2 + C.$$

so $y(t) = \dfrac{-1}{\frac{1}{2}t^2 + C}$ $\qquad -\frac{1}{C} = y(0) = 1$ so $C = -1$.

$$y(t) = \frac{-1}{\frac{1}{2}t^2 - 1}$$

(ii) interval of existence is $(-\sqrt{2}, \sqrt{2})$.

(iii) yes because $\frac{\partial}{\partial y}(ty^2) = 2ty$ is continuous.

(b) integrating factor $u = e^{t^2}$.

$$\frac{d}{dt}(e^{t^2}y) = e^{t^2}e^{-t^2}\sin t = \sin t. \Rightarrow e^{t^2}y = -\cos t + C.$$

so

$$y(t) = -e^{-t^2}\cos t + Ce^{-t^2}.$$

[10 pts]

2. Find the integral curve of

$$\alpha = (\sin(x+y) + 2x)dx + (y^2 + \sin(x+y))dy$$

that passes through the point $(\pi, 0)$. (You may define this curve implicitly.)

(Fall 2008: not responsible for this)

let $f(x,y) = -\cos(x+y) + x^2 + \frac{y^3}{3}$.

then

$\alpha = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df$ so is exact.

Integral curves are given by $f(x,y) = \text{const}$.

$f(\pi,0) = -\cos(\pi) + \pi^2 = \pi^2 - 1$

so the integral curve through $(\pi,0)$ is defined implicitly by $-\cos(x+y) + x^2 + \frac{y^3}{3} = \pi^2 - 1$.

[15 pts]

3. (a) Solve the initial value problem :

$$y'' + 4y = 2\cos(2t), \quad y(0) = 0, y'(0) = 0.$$

(b) Solve the initial value problem :

$$y'' + 4y = 2\cos(2t), \quad y(0) = 1, y'(0) = -1.$$

(a) take Laplace transform

$$s^2 Y(s) + 4Y(s) = 2 \cdot \frac{s}{s^2+4}$$

$$Y(s) = \frac{2s}{(s^2+4)^2} = \frac{d}{ds}\frac{1}{s^2+4}$$

so $y(t) = -t\,\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = -\frac{t}{2}\,\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = -\frac{t}{2}\sin(2t).$

(b). General solution to homogeneous eqn $y'' + 4y = 0$

char eqn is $\lambda^2 + 4 = 0$ so $\lambda = \pm 2i$.

general homog. solution is $c_1\cos(2t) + c_2\sin(2t)$.

General sol^n to $y'' + 4y = 2\cos(2t)$ is

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{t}{2}\sin(2t).$$

Find c_1, c_2: $y(0) = c_1$ so $c_1 = 1$.

$y'(0) = 2c_2$ so $c_2 = -\frac{1}{2}$.

Answer is $y(t) = \cos(2t) - \frac{1}{2}\sin(2t) - \frac{t}{2}\sin(2t).$

[15 pts]

4. Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

   (a) Find the matrix exponential e^{At}.

   (b) Solve the initial value problem :

$$\mathbf{y}' = A\mathbf{y} + \begin{pmatrix} e^{2t}\sin(t) \\ e^{2t} \\ 0 \end{pmatrix}, \qquad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Note that A has eigenvalue 2 with algebraic multiplicity 3.

$e^{At} = e^{(A-2Id)t + 2Id\,t}$. $A-2Id$ & $2Id$ commute so we can write:

$$e^{At} = e^{(A-2Id)t} \cdot e^{2Id\,t} = e^{2t} \cdot e^{(A-2Id)t}.$$

$$= e^{2t} \cdot \left(Id + \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{t^2}{2} + \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 \frac{t^3}{3!} + \cdots \right)$$

$$= e^{2t}\left(Id + \begin{pmatrix} 0 & 2t & t \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{t^2}{2} + 0 + 0 + \cdots \right)$$

$$= e^{2t} \begin{pmatrix} 1 & 2t & t+t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Variation of parameters. $\underline{v}' = e^{-At} \begin{pmatrix} e^{2t}\sin t \\ e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -2t & -t+t^2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin t \\ 1 \\ 0 \end{pmatrix}$

so $\underline{v}' = \begin{pmatrix} \sin t - 2t \\ 1 \\ 0 \end{pmatrix}$ so $\underline{v}(t) = \begin{pmatrix} -\cos t - t^2 + c_1 \\ t + c_2 \\ c_3 \end{pmatrix}$. want $\underline{v}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

so $c_1 = 1$, $c_2 = 0$, $c_3 = 1$

$$y(t) = e^{2t} \begin{pmatrix} 1 & 2t & t+t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \cos t - t^2 \\ t \\ 1 \end{pmatrix}.$$

[15 pts]

5. (a) Sketch the phase portrait for $\mathbf{y}' = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \mathbf{y}$.

(b) Determine whether $\mathbf{0}$ is asymptotically stable, stable or unstable.

(a) Find eigenvalues of A: $\cancel{A}\ (-1-\lambda)(-1-\lambda)-4$

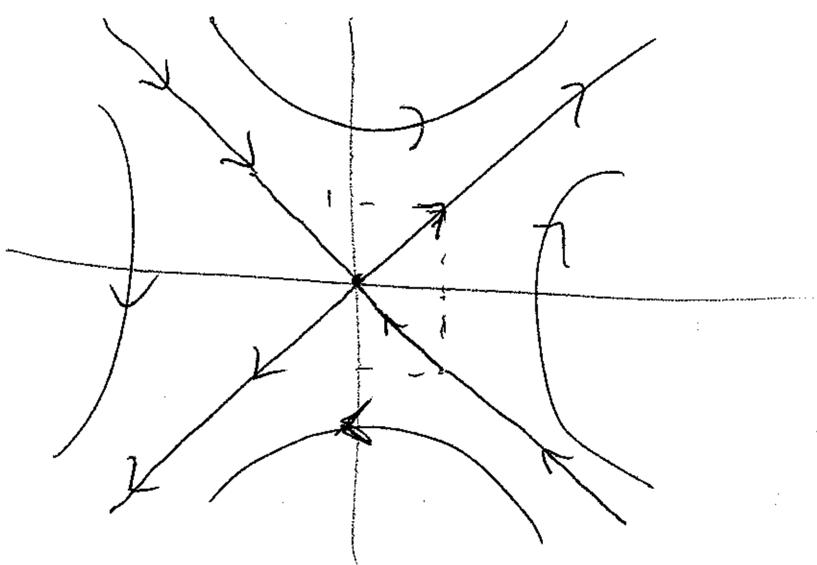
$= \lambda^2+2\lambda+1-4 = \lambda^2+2\lambda-3. = (\lambda-1)(\lambda+3).$

$\Rightarrow \lambda=1, \lambda=-3$ eigenvalues.

$\Rightarrow$ is a saddle.

Find eigenvectors given by: for $\lambda=1$: $\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right) \rightsquigarrow$ eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for $\lambda=-3$ $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \rightsquigarrow$ eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.



(b) One eigenvalue is positive $\Rightarrow$ <u>unstable</u>.

[15 pts]

6. Consider the nonlinear system of differential equations given by :

$$x' = 2x(2-x) \quad = 4x - 2x^2 .$$
$$y' = y(1+x^2)$$

(a) Find the equilibrium solutions.

(b) For each equilibrium solution you found in (a), determine its stability.

(a) $\begin{cases} 2x(2-x) = 0 \\ y(1+x^2) = 0 \end{cases} \Rightarrow \begin{cases} x=0 \text{ or } x=2. \\ y=0 \text{ or } 1+x^2=0. \end{cases}$ x real so $1+x^2 \neq 0$.

2 eqm solutions: $(0,0)$ and $(2,0)$.

(b) Compute $Df(x,y) = \begin{pmatrix} 4-4x & 0 \\ 2xy & 1+x^2 \end{pmatrix}$.

- $Df(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ $\rightsquigarrow$ eigenvalues are 4 and 1, $\Rightarrow$ is a source $\Rightarrow$ unstable.

- $Df(2,0) = \begin{pmatrix} -4 & 0 \\ 0 & 5 \end{pmatrix}$ $\longrightarrow$ eigenvalues are -4 and 5. $\Rightarrow$ saddle $\Rightarrow$ unstable.

[15 pts]

7. (a) Find the Laplace transform of the solution to the initial value problem

$$y'' + 2y' - y = \cos(2t), \qquad y'(0) = 1, y(0) = -1.$$

(You do **not** need to solve for $y(t)$.)

(b) Find the inverse Laplace transform of $\frac{-4}{(s-1)^2(s-3)}$.

The two parts of this question are unrelated.

(a) $\mathcal{L}(y') = sY(s) + 1$

$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + s - 1.$

So

$s^2Y(s) + s - 1 + 2sY(s) + 2 - Y(s) = \frac{s}{s^2+4}$

$Y(s)\cdot(s^2+2s-1) = \frac{s}{s^2+4} - s - 1.$

$Y(s) = \frac{s}{(s^2+4)(s^2+2s-1)} - \frac{1+s}{(s^2+2s-1)}$

(b) $\frac{-4}{(s-1)^2(s-3)} = \frac{a}{s-1} + \frac{b}{(s-1)^2} + \frac{c}{s-3}$ Partial Fractions.

$-4 = a(s-1)(s-3) + b(s-3) + c(s-1)^2.$

$-4 = as^2 - 4as + 3a + bs - 3b + cs^2 - 2cs + c$

$-4 = (a+c)s^2 + (-4a+b-2c)s + (3a-3b+c.)$

$a+c=0 \qquad -4a-2c+b=0 \qquad 3a-3b+c=-4.$

Problem 7b continued

$$\begin{cases} a=-c. \\ b=4a+2c=2a. \\ c=-4-3a+3b=-4-3a+6a=-4+3a=-4-3c. \end{cases}$$

$$\begin{cases} a=-c \\ b=2a \\ 4c=-4 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=2. \\ c=-1. \end{cases}$$

$$\frac{-4}{(s-1)^2(s-3)} = \frac{1}{s-1} + \frac{2}{(s-1)^2} - \frac{1}{s-3}.$$

$$\mathcal{L}^{-1}\left(\frac{-4}{(s-1)^2(s-3)}\right) = e^t + 2\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) - e^{3t}.$$

Now $\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) = e^t \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = e^t \cdot t.$

thus

$$\mathcal{L}^{-1}\left(\frac{-4}{(s-1)^2(s-3)}\right) = e^t + 2te^t - e^{3t}.$$

8. Consider the nonlinear system of differential equations given by : [a–c: 10 pts]

$$x' = 2y$$
$$y' = -2x - 4x^3$$

(a) Show that $(0,0)$ is the only equilibrium solution.

(b) What does the linearization at $(0,0)$ tell us about stability?

(c) Suppose that $(x(t), y(t))$ is a solution to the nonlinear system. Show that it is an integral curve of

$$\alpha = (2x + 4x^3)dx + 2ydy.$$

(d) [Bonus – 10 pts] Use the fact that α is exact to conclude that $(0,0)$ is stable, but not asymptotically stable.

(a) Eqm sol^n: $\begin{cases} 2y = 0 \\ -2x - 4x^3 = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ 0 = -2(x + x^2) = -2x(1+x^2). \end{cases}$

x real so $1+x^2 \neq 0$ so no other eqm points other than $(0,0)$.

(b) $Df(x,y) = \begin{pmatrix} 0 & 2 \\ -2 - 12x^2 & 0 \end{pmatrix}$ so $Df(0,0) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

eigenvalues are roots of $\lambda^2 + 4 \Rightarrow \lambda = \pm 2i$.

These are purely imaginary so the linearization tells us <u>nothing</u>.

(c) (Fall 2008: not responsible for this. Instead see alternate problem, next page.)

Suffices to compute $(2x + 4x^3)x' + 2yy' = 2y(2x + 4x^3) + 2y(-2x - 4x^3) = 0$

so is integral curve of α.

$\longrightarrow$

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(d) let $f(x,y) = x^2 + x^4 + y^2$. then $\alpha = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

so f is constant on solutions $(x(t), y(t))$.

Thus, $(x(t), y(t))$ stays on a level set of f.

Level sets of f are of the form

$$x^2 + y^2 + x^4 = c.$$

Notice that $x^2+y^2 \leq x^2+y^2+x^4$ so the level set $x^2+y^2+x^4=c$ is contained inside the disk of radius $\sqrt{c}$.

Level sets then look like:

, at least near 0.

Thus any orbit that starts near 0, stays near 0. $\Rightarrow$ 0 is stable.

However, 0 can't be asymptotically 0, because if $f(x_0, y_0) = c_0$,

$x^2+y^2+x^4 = c_0$ for all time so we <u>can't</u> have $\lim_{t \to +\infty} (x(t), y(t)) = 0$.

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Alternate questions: (Fall 2008)

(c) Show $h(x,y) = x^2 + y^2 + x^4$ is a conserved quantity.

(d) Using this, show that $(0,0)$ is stable but not asymptotically stable.

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