Stanford University Department of Mathematics

Math 53 Final

Instructor: Gautam Iyer

March 16, 2009
Duration: 3 hours

Please **PRINT** your name below

Last Name: IYER First & Middle Names: GAUTAM

Circle the name of your TA

Dominic Dotterrer Daniel Mathews Tania Rojas-Esponda

Sign below to acknowledge and accept the Honor Code.

DO NOT OPEN THIS TEST UNTIL INSTRUCTED TO DO SO.

Instructions:

- No electronic devices of any kind (e.g. calculators, computers, cell-phones) are allowed. This is a closed book closed notes text.
- You may quote theorems from your textbook or from class if you make an appropriate reference.
- ~~Show all your work. *Correct solutions with omitted steps will be penalised.*~~
- A few pages in this exam have been left blank for scratch work. Should you need more space, blank sheets will be provided on request. These must be stapled to your exam when you finish.
- This exam will bring you bad karma if written on for more than 3 hours. Bad karma sometimes manifests itself in the form of a 1,000,000 point grading error against your favor.
- The questions on this exam are divided into three parts. The first part involves of short, standard computations (i.e. easy). The second part involves of slightly longer computations and is a bit of a workout (i.e. harder). The third part makes literal the Shakespearian quote '*Beware the ides of March*'.
- Good luck, and have a nice break. Your graded exams can be picked up from your TA next quarter.

Question	Points
1	
2	
3	
4	
5	
6	
7	
8	
Total [85]	

Part 1: Menlo Park Community College

$\boxed{10}$ 1. Does the system

$$\frac{dx}{dt} = 2xy + e^x$$
$$\frac{dy}{dt} = x^2 - y^2$$

have a closed trajectory? Justify. [By closed trajectory, we mean a non-constant solution of the system who's parametric representation is a closed curve.]

Let $\vec{F}(x,y) = \begin{pmatrix} 2xy + e^x \\ x^2 - y^2 \end{pmatrix}$. $\nabla \cdot \vec{F} = 2y + e^x - 2y = e^x$

$\nabla \cdot \vec{F}$ always > 0

$\Rightarrow$ No closed trajectories.

$\boxed{10}$ 2. Find the general solution of the ODE $\frac{dy}{dx} = x(2+xy)$. Find also the interval of existence of solutions to this ODE with initial data $y(0) = 1$.

$$\frac{dy}{dx} = x(2+xy) \iff \frac{dy}{dx} - x^2 y = 2x$$

Integrating factor $\mu(x) = e^{\int -x^2 dx} = e^{-x^3/3}$

$$\Rightarrow \frac{d}{dx}(\mu y) = 2x\mu \Rightarrow e^{-x^3/3} y = \int 2x e^{-x^3/3} dx$$

= Oops! something you can't integrate

$\therefore$ general sol is $\boxed{y = e^{x^3/3} \int 2x e^{-x^3/3} dx + Ce^{x^3/3}}$ or $\boxed{y = e^{x^3/3} \int_0^x 2s e^{-s^3/3} ds + Ce^{x^3/3}}$

The equation is linear with continuous coefficients

$\Rightarrow$ Interval of existence $= (-\infty, \infty)$ (no matter what a is)

$\boxed{10}$ 3. Let $A = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$. Find the general solution of the system $\frac{d\vec{x}}{dt} = A\vec{x}$, and sketch a phase portrait.

① Evalues: The characteristic polynomial is

$$f(\lambda) = \lambda^2 - Tr(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

$\Rightarrow$ Evalues $= 1$

② E vectors: $N(A - I) = N\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find $\vec{v}_2$ so that $A\vec{v}_2 = \vec{v}_2 + \vec{v}_1$

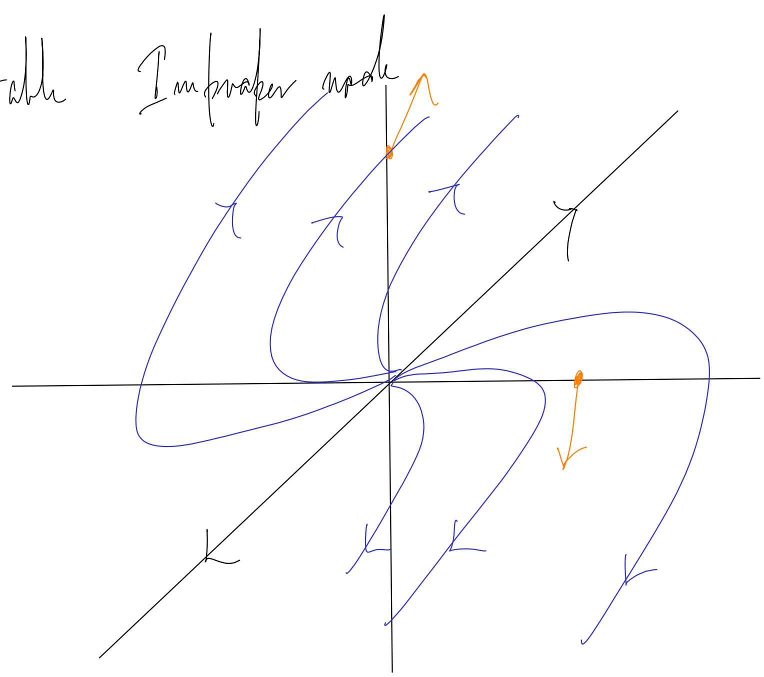
$\Leftrightarrow (A - I)\vec{v}_2 = \vec{v}_1 \Leftrightarrow \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Choose $\vec{v}_2 = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$.

Now the general solution is given by

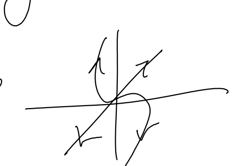
$$\vec{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \left[\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$$

$$= c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} t \\ 1/2 + t \end{pmatrix}$$

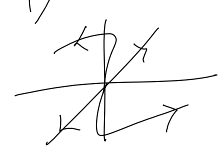
Unstable Improper node



Compute $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$
& $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
(the orange arrows) to determine which way the node is oriented
(i.e.



instead of



)

Part 2: Stanford University

4. Let b, c be constants, and consider the ODE

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \tag{1}$$

$\boxed{5}$ (a) Find a condition on b and c which will guarantee that solutions of the ODE (1) exhibit exponentially damped oscillations as $t \to \infty$. [By exponentially damped oscillations, we mean $x(t)$ is of the form $e^{-\alpha t}g(t)$ where g is periodic and $\alpha > 0$.]

$\boxed{5}$ (b) Suppose x_1 and x_2 are two solutions of the ODE (1). Let $W(t) = x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}$. Find $W(t)$ explicitly as a function of $W(0)$, t, b and c. [HINT: Compute $\frac{dW}{dt}$, and express your answer in terms of W, b and c. *This part is* ***unrelated*** *to the previous subpart.*]

(a) Put $z = \frac{dy}{dt}$. Then $\begin{pmatrix} dy/dt \\ dz/dt \end{pmatrix} = \begin{pmatrix} z \\ -bz - cy \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}}_{A} \begin{pmatrix} y \\ z \end{pmatrix}$

$\left.\begin{array}{l} \mathrm{Tr}(A) = -b \\ \det(A) = c \end{array}\right\}$ Char poly (A) is $\lambda^2 + b\lambda + c$

Evalues of (A) are $-b \pm \sqrt{b^2 - 4c}$.

General is $e^{-\alpha t}(\sin() \text{ or } \cos()) \iff$ Complex evalues with negative real part.

$\therefore$ We have exponentially damped osc. $\iff \boxed{b < 0 \ \& \ b^2 - 4c < 0}$

(b) $\frac{dW}{dt} = x_1 \frac{d^2x_2}{dt^2} + \frac{dx_1}{dt}\frac{dx_2}{dt} - \left(\frac{dx_1}{dt}\frac{dx_2}{dt} + x_2\frac{d^2x_1}{dt^2}\right)$

$= x_1\left(-b\frac{dx_2}{dt} - cx_2\right) - x_2\left(-b\frac{dx_1}{dt} - cx_1\right) = -b\left(x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}\right)$

$= -bW$

$\Rightarrow \frac{dW}{dt} = -bW \Rightarrow \boxed{W(t) = e^{-bt}W(0)}$

$\boxed{15}$ 5. Suppose x and y satisfy the equations

$$\frac{dx}{dt} = x(8 - 3x - 2y)$$
$$\frac{dy}{dt} = y(8 - x - 6y)$$

Find all critical points of this system, and determine the stability and type (i.e. stable node, unstable spiral, etc.) of each of these critical points. Use this to draw an educated guess of the phase portrait of this system.

Let $\vec{F}(x,y) = \begin{pmatrix} x(8-3x-2y) \\ y(8-x-6y) \end{pmatrix}$. Then Jacobian$(\vec{F}) = \begin{pmatrix} 8-6x-2y & -2x \\ -y & 8-x-12y \end{pmatrix}$

Critical points:

(1) $\boxed{x=0,\ y=0}$: Jacobian $= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$. Diagonal. Evalues $= 8, 8$

$\boxed{\Rightarrow \text{unstable node}}$

(2) $x=0$, $8-x-6y=0 \Leftrightarrow \boxed{x=0 \ \& \ y=4/3}$

Jacobian $= \begin{pmatrix} 8-8/3 & 0 \\ -4/3 & 8-16 \end{pmatrix} = \begin{pmatrix} 16/3 & 0 \\ -4/3 & -8 \end{pmatrix}$ Triangular evalues $16/3$ & -8

$\Rightarrow \boxed{\text{saddle (unstable)}}$. Also $\begin{pmatrix} 16/3 & 0 \\ -4/3 & -8 \end{pmatrix} \vec{e}_2 = -8\vec{e}_2$, so stable direction is along $\vec{e}_2$

(3) $y=0$ & $8-3x-2y=0 \Leftrightarrow \boxed{x = 8/3 \ \& \ y=0}$

Jacobian $= \begin{pmatrix} 8-16 & -16/3 \\ 0 & 8-8/3 \end{pmatrix} = \begin{pmatrix} -8 & -16/3 \\ 0 & 16/3 \end{pmatrix}$ Triangular. evalues -8 & $16/3$

$\Rightarrow \boxed{\text{saddle (unstable)}}$. Also $\begin{pmatrix} -8 & -16/3 \\ 0 & 16/3 \end{pmatrix} \vec{e}_1 = -8\vec{e}_1 \Rightarrow$ stable direction is along $\vec{e}_1$

(4) $8-3x-2y=0$ & $8-x-6y=0$

$\Leftrightarrow \begin{pmatrix} 3 & 2 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{18-2}\begin{pmatrix} 6 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 8 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 6-2 \\ -1+3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $\therefore$ $\boxed{x=2 \ \& \ y=1}$

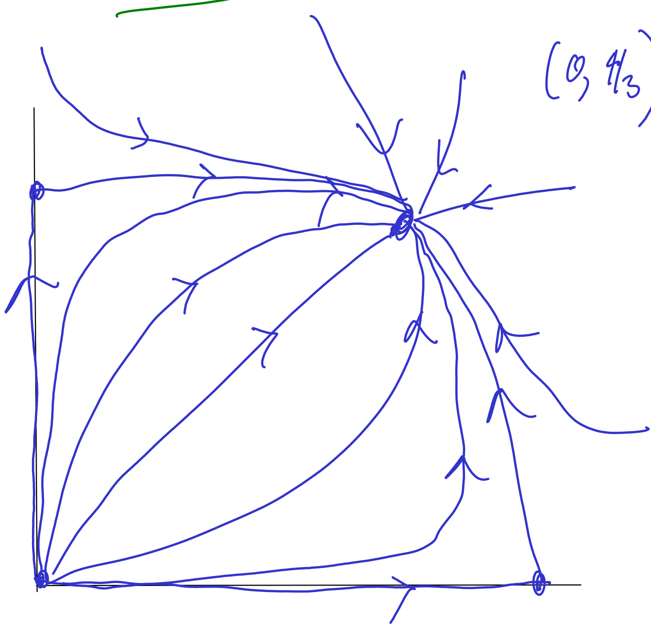
Jacobian: $\begin{pmatrix} 8-12-2 & -4 \\ -1 & 8-2-12 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ -1 & -6 \end{pmatrix}$

Trace $= -12$ determinant $= 36-4 = 32 > 0$

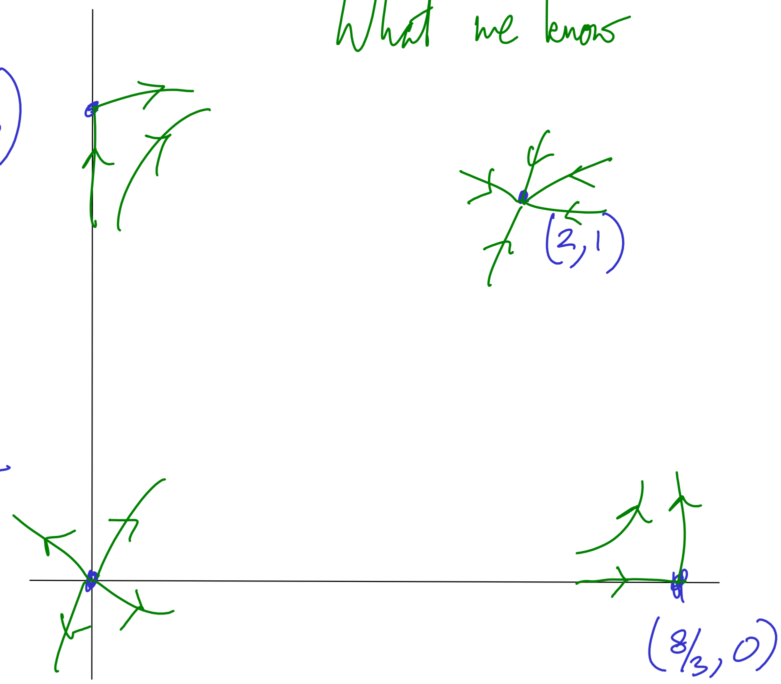
$\text{Trace}^2 - 4(\text{determinant}) = 12^2 - 4(32) = 144 - 128 = 16 > 0$

$\Rightarrow$ Real negative evalues $\left(\dfrac{-12 \pm 4}{2}\right) \Rightarrow$ $\boxed{\text{Stable node}}$

Guess



What we know



The first Midterm and many tasteless bowls of Ramen Noodles™ later, we now embark on a quest to cleverly convert our bowl of Ramen Noodles™ to ashes. Some would argue that this doesn't affect the edibility...

10 6. You place a bowl of Ramen Noodles™ on a stove, which *supplies heat at a constant rate*. The bowl of Ramen Noodles™ also *loses heat to the surroundings at a rate proportional to the temperature difference*. Assume that the room temperature is 25°C, and that the bowl of Ramen Noodles™ is initially at room temperature. Ten minutes after turning on the stove, your Ramen Noodles™ are at 45°C. Twenty minutes after turning on the stove the Ramen Noodles™ are at 64°C. What will be the temperature of the Ramen Noodles after a very long time (i.e. as $t \to \infty$)? [HINT: The **two** italicized statements will help you set up an ODE for the temperature of the Ramen Noodles™. The remainder of the given info will help you determine the unknown coefficients in the ODE. If you do the math correctly, you will get a 'nice answer', and a final temperature which is certainly hot enough to incinerate any bowl of Ramen Noodles™.]

Let θ = temperature of noodles. T_0 = room temp (25°C), α, β constants.

$$\frac{d\theta}{dt} = \text{rate of heating (from stove)} - \text{rate of heat loss (to environment)} = \beta - \alpha(\theta - T_0)$$

$$\Rightarrow \frac{d\theta}{\beta - \alpha(\theta - T_0)} = dt \Rightarrow \left.\frac{-1}{\alpha}\ln\left(\beta - \alpha(\theta - T_0)\right)\right]_0^t = t$$

$$\Rightarrow \ln\left(\beta - \alpha(\theta - T_0)\right) - \ln\left(\beta - \alpha(T_0 - T_0)\right) = -\alpha t$$

$$\Rightarrow \ln\left(1 - \frac{\alpha}{\beta}(\theta - T_0)\right) = -\alpha t \Rightarrow \boxed{\theta = \frac{\beta}{\alpha}\left(1 - e^{-\alpha t}\right) + T_0}$$

Note that as $t \to \infty$, $\theta \to \beta/\alpha + T_0$. So we need to find β/α.

Now plug in data: (1) $t = 10 \Rightarrow \ln\left(1 - \frac{\alpha}{\beta}(45 - 25)\right) = -10\alpha$

(2) $t = 20 \Rightarrow \ln\left(1 - \frac{\alpha}{\beta}(64 - 25)\right) = -20\alpha$

Dividing (2) by (1) gives $2 = \dfrac{\ln\left(1 - \frac{\alpha}{\beta}(39)\right)}{\ln\left(1 - \frac{\alpha}{\beta}(20)\right)} \Rightarrow \ln\left(1 - \frac{\alpha}{\beta}(39)\right) = 2\ln\left(1 - \frac{\alpha}{\beta}(20)\right)$

$$\Rightarrow 1 - 39\frac{\alpha}{\beta} = \left(1 - \frac{20\alpha}{\beta}\right)^2 = 1 + \frac{400\alpha^2}{\beta^2} - \frac{40\alpha}{\beta}$$

$$\Rightarrow \frac{\alpha}{\beta} = 400\frac{\alpha^2}{\beta^2} \Rightarrow \beta/\alpha = 400$$

So our maximum temperature is $T_0 + \beta/\alpha = 25 + 400 = \boxed{425}$

Part 3: Iyer University

$\boxed{5}$ 7. (a) Compute the inverse Laplace transform of the function $F(s) = \frac{1}{(1+s^2)^2}$. [HINT: Express F as a linear combination of $L(\sin t)$, $L(\cos t)$, $\frac{d}{ds}L(\sin t)$ and $\frac{d}{ds}L(\cos t)$.]

$\boxed{5}$ (b) Solve the (second order) ODE $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^t \sin t$, with initial data $y(0) = a$ and $\frac{dy}{dt}\big|_{t=0} = b$.

(a) $\frac{d}{ds}L(\cos t) = \frac{d}{ds}\left(\frac{s}{1+s^2}\right) = \frac{1+s^2 - s(2s)}{(1+s^2)^2} = \frac{1-s^2}{(1+s^2)^2} = \frac{2-(s^2+1)}{(1+s^2)^2} = \frac{2}{(1+s^2)^2} - \frac{1}{1+s^2}$

$\therefore \frac{1}{(1+s^2)^2} = \frac{1}{2}\left(\frac{d}{ds}(L(\cos t)) + \frac{1}{1+s^2}\right) = \frac{1}{2}\left(L(-t\cos t) + L(\sin t)\right)$

$$\boxed{\Rightarrow L^{-1}\left(\frac{1}{(1+s^2)^2}\right) = \frac{1}{2}\left(\sin t - t\cos t\right)}$$

(b) Let $Y = Ly$: $L\left(\frac{d^2y}{dt^2}\right) = s^2Y(s) - as - b$ & $L\left(\frac{dy}{dt}\right) = sY(s) - a$

$L(e^t \sin t) = \frac{1}{(s-1)^2+1}$ $\therefore$ Taking L of both sides of an ODE gives

$$s^2Y - as - b - 2sY - 2a + 2Y = \frac{1}{(s-1)^2+1}$$

$\Rightarrow (s^2 - 2s + 2)Y = \frac{1}{(s-1)^2+1} + as + 2a + b \Rightarrow Y = \frac{1}{((s-1)^2+1)^2} + \frac{as}{(s-1)^2+1} + \frac{2a+b}{(s-1)^2+1}$

(1) $L^{-1}\left(\frac{1}{((s-1)^2+1)^2}\right) = e^t L^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \frac{e^t}{2}(\sin t - t\cos t)$

(2) $L^{-1}\left(\frac{as}{(s-1)^2+1} + \frac{2a+b}{(s-1)^2+1}\right) = L^{-1}\left(\frac{a(s-1)}{(s-1)^2+1} + \frac{3a+b}{(s-1)^2+1}\right)$

$= ae^t\cos t + (3a+b)e^t\sin t$

$$\therefore \boxed{y = L^{-1}(Y) = \frac{e^t}{2}\left(\sin t - t\cos t + a\cos t + (3a+b)\sin t\right)}$$

$\boxed{10}$ 8. Let V be a scalar function of two variables (i.e. the domain of V is $\mathbb{R}^2$, and the range is $\mathbb{R}$) such that the point $(0,0)$ is a (strict) local minimum of V. Consider the autonomous system of ODE's

$$\left.\begin{aligned} \frac{dx}{dt} &= -\frac{\partial V}{\partial x}(x,y) \\ \frac{dy}{dt} &= -\frac{\partial V}{\partial y}(x,y). \end{aligned}\right\} \tag{2}$$

Since $(0,0)$ is a local minimum of V, we know $\frac{\partial V}{\partial x}(0,0) = \frac{\partial V}{\partial y}(0,0) = 0$. Thus the origin $(0,0)$ is a critical point of the system of equations (2). Determine the local stability of this critical point, and justify your answer. [WARNING: Any proof involving second derivatives of V is almost certainly incorrect, and will get almost no partial credit. The hint is to compute the derivative of V along the parametric curve representing the solution of the system (2) (we did a similar computation in class, though in a different context). You may assume that for all points close to $(0,0)$, $\nabla V \neq \vec{0}$. Please also make sure you CLEARLY justify your answer – getting the stability correct, but not providing any justification will be worth almost nothing.]

Say $x(t)$, $y(t)$ is a solution of this system:

$$\frac{d}{dt}\Big(V(x(t), y(t))\Big) = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = -\left(\frac{\partial V}{\partial x}\right)^2 - \left(\frac{\partial V}{\partial y}\right)^2 < 0$$

$\Rightarrow$ along trajectories of this system, V is decreasing.

$\Rightarrow$ Expect $(x(t), y(t)) \longrightarrow$ a minimum of V as $t \to \infty$

Great! $(0,0)$ is a minimum of V. $\Rightarrow$ $\boxed{\text{locally stable}}$

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Table 1: Elementary Laplace transforms, and properties

f(t)	F(s)	Notes
t^n	$\dfrac{n!}{s^{n+1}}$	$s > 0$, $n = 0, 1, 2\ldots$, and remember $0! = 1$.
e^{at}	$\dfrac{1}{s-a}$	$s > a$, $a \in \mathbb{R}$ (or $s > \mathrm{Re}(a)$ & $a \in \mathbb{C}$)
$\sin(at)$	$\dfrac{a}{s^2+a^2}$	$s > 0$, $a \in \mathbb{R}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$	$s > 0$, $a \in \mathbb{R}$
$e^{at}f(t)$	$F(s-a)$	$a \in \mathbb{R}$, $F = Lf$, $s - a \in \mathrm{Domain}(F)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - {}$ $- s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$	$F = Lf$, $f^{(n)} = \frac{d^n}{dt^n} f$, $f^{(0)} = f$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$F = Lf$, $F^{(n)} = \frac{d^n}{ds^n} F$