

# Math53: Ordinary Differential Equations Autumn 2004

## Final Exam Solutions

### Problem 1 (25pts)

Find the solution  $y=y(t)$ , including the interval of existence, to the initial value problem

$$(t^2+1)y' + 4ty = 8t, \quad y(0) = 1.$$

This ODE can be treated as either linear or separable. If treated as linear:

$$\begin{aligned} y' + \frac{4t}{t^2+1}y &= \frac{8t}{t^2+1} \implies P(t) = e^{\int \frac{4t}{t^2+1} dt} = e^{2\ln(t^2+1)} = (t^2+1)^2 \\ \implies (t^2+1)^2 \left( y' + \frac{4t}{t^2+1}y \right) &= (t^2+1)^2 \frac{8t}{t^2+1} \\ \implies (t^2+1)^2 y' + 4t(t^2+1)y &= 8t(t^2+1) \implies ((t^2+1)^2 y)' = 8t(t^2+1) \\ \implies (t^2+1)^2 y(t) &= \int 8t(t^2+1) dt = 2(t^2+1)^2 + C. \end{aligned}$$

Applying the initial conditions, we find  $C$ :

$$(0^2+1)^2 \cdot 1 = 2(0^2+1)^2 + C \implies C = -1 \implies \boxed{y(t) = 2 - (t^2+1)^{-2}}$$

If treated as separable:

$$\begin{aligned} (t^2+1) \frac{dy}{dt} &= 4t(2-y) \implies \frac{dy}{2-y} = \frac{4t dt}{t^2+1} \implies \int \frac{dy}{(2-y)} = \int \frac{4t dt}{t^2+1} \\ \implies -\ln|2-y(t)| &= 2\ln(t^2+1) + C \end{aligned}$$

Applying the initial conditions, we find  $C$ :

$$\begin{aligned} -\ln|2-1| &= 2\ln(0^2+1) + C \implies C = 0 \implies \ln|2-y(t)| = -2\ln(t^2+1) \\ \implies |2-y(t)| &= (t^2+1)^{-2} \implies 2-y(t) = (t^2+1)^{-2} \\ \implies \boxed{y(t) = 2 - (t^2+1)^{-2}} \end{aligned}$$

Note that we need to use the initial condition to get rid of the absolute value above.

Since the solution is defined for all  $t$ , the interval of existence is  $\boxed{(-\infty, \infty)}$

**Problem 2 (20pts)**

Find all solutions  $y=y(t)$  to the ODE

$$y' = y(y-3)t^2$$

explicitly.

This ODE is linear:

$$\begin{aligned}\frac{dy}{dt} = y(y-3)t^2 &\implies \frac{dy}{y(y-3)} = t^2 dt \implies \int \frac{1}{3} \left( \frac{1}{y-3} - \frac{1}{y} \right) dy = \int t^2 dt \\ &\implies \ln |y-3| - \ln |y| = t^3 + C \implies \ln \left| \frac{y-3}{y} \right| = t^3 + C \\ &\implies \left| 1 - \frac{3}{y} \right| = Ae^{t^3} \quad A > 0 \implies 1 - \frac{3}{y(t)} = Ae^{t^3} \quad A \neq 0 \\ &\implies y(t) = \frac{3}{1 - Ae^{t^3}} \quad A \neq 0.\end{aligned}$$

However, since we divided by  $y(y-3)$ , we might have missed the constant solutions  $y(t) = 0$  and  $y(t) = 3$ . The latter corresponds to  $A=0$ . Thus, the general solution is

$y(t) = \frac{3}{1 - Ae^{t^3}} \quad A \in \mathbb{R}, \quad y(t) = 0$
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### Problem 3 (30pts)

Find the solution  $y=y(t)$  to the initial value problem

$$y'' - 5y' + 6y = 2te^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Since the homogeneous solutions and the forcing terms *both* involve exponentials, the easiest approach is likely to be via the Laplace Transform. Let  $Y=Y(s)$  be the Laplace Transform of  $y=y(t)$ . Using the two LT tables, we obtain

$$\begin{aligned} \mathcal{L}y' &= s\mathcal{L}y - y(0) = sY, \quad \mathcal{L}y'' = s\mathcal{L}y' - y'(0) = s^2Y \quad \implies \\ y'' - 5y' + 6y &= 2te^{3t} \quad \implies \quad (s^2 - 5s + 6)Y = \frac{2}{(s-3)^2} \quad \implies \quad Y(s) = \frac{2}{(s-2)(s-3)^3}. \end{aligned}$$

We next apply quick PFs:

$$\begin{aligned} \frac{1}{(s-2)(s-3)} &= \frac{1}{-2 - (-3)} \left( \frac{1}{s-3} - \frac{1}{s-2} \right) = \frac{1}{s-3} - \frac{1}{s-2} \quad \implies \\ Y(s) &= \frac{2}{(s-3)^2} \cdot \frac{1}{(s-2)(s-3)} = \frac{2}{(s-3)^2} \cdot \left( \frac{1}{s-3} - \frac{1}{s-2} \right) = \frac{2}{(s-3)^3} - \frac{2}{(s-3)} \cdot \left( \frac{1}{s-3} - \frac{1}{s-2} \right) \\ &= \frac{2}{(s-3)^3} - \frac{2}{(s-3)^2} + \frac{2}{s-3} - \frac{2}{s-2} \quad \implies \quad \boxed{y(t) = t^2e^{3t} - 2te^{3t} + 2e^{3t} - 2e^{2t}} \end{aligned}$$

by the first LT table.

The more standard approach is to first find the general solution to the ODE as  $y=y_h+y_p$  and then to use the initial conditions to find the two constants. The characteristic polynomial in this case is

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Its roots are  $\lambda_1 = 2$  and  $\lambda_2 = 3$ . Thus, the general solution to the associated homogeneous equation is

$$y_h(t) = C_1e^{2t} + C_2e^{3t}.$$

We next use the method of undetermined coefficients to find a particular solution  $y_p = y_p(t)$  to the inhomogeneous equation. Since the forcing term is of the form  $Ate^{3t}$ , we would normally try  $y_p(t) = Bte^{3t} + Ce^{3t}$ . However, since  $e^{3t}$  is a solution to the associated homogeneous equation, we instead must try  $y_p(t) = Bt^2e^{3t} + Cte^{3t}$ :

$$\begin{aligned} y_p &= Bt^2e^{3t} + Cte^{3t} \quad \implies \quad y_p' = 3Bte^{3t} + (2B+3C)te^{3t} + Ce^{3t} \\ &\implies \quad y_p'' = 9Bte^{3t} + (12B+9C)te^{3t} + (2B+6C)e^{3t} \quad \implies \\ (9Bt^2 + (12B+9C)t + (2B+6C))e^{3t} &- 5(3Bt^2 + (2B+3C)t + C)e^{3t} + 6(Bt^2 + Ct)e^{3t} = 2te^{3t} \\ &\implies \quad 2Bt + (2B+C) = 2t \iff \begin{cases} 2B = 2 \\ 2B + C = 0 \end{cases} \iff \begin{cases} B = 1 \\ C = -2 \end{cases} \\ &\implies \quad y_p(t) = t^2e^{3t} - 2te^{3t} \quad \implies \quad y(t) = C_1e^{2t} + C_2e^{3t} + t^2e^{3t} - 2te^{3t}. \end{aligned}$$

Finally, we use the initial conditions to determine  $C_1$  and  $C_2$ :

$$\begin{cases} y(0) = C_1 + C_2 = 0 \\ y'(0) = 2C_1 + 3C_2 - 2 = 0 \end{cases} \quad \implies \quad C_1 = -2, \quad C_2 = 2 \quad \implies \quad \boxed{y(t) = -2e^{2t} + 2e^{3t} + t^2e^{3t} - 2te^{3t}}$$

**Problem 4 (30pts)**

(a; 5pts) Verify that  $y_1(t)=t$  and  $y_2(t)=t^{-1}$  are linearly independent solutions of the ODE

$$t^2 y'' + t y' - y = 0, \quad y = y(t).$$

(b+c; 9+16pts) Find the general solution to the ODE

$$t^2 y'' + t y' - y = 8t^2 e^{2t}, \quad y = y(t).$$

(a) We plug in  $y_1(t)=t$  and  $y_2(t)=t^{-1}$  into the homogeneous ODE:

$$\begin{aligned} t^2 y_1'' + t y_1' - y_1 &= t^2 \cdot 0 + t \cdot 1 - t = 0 & \checkmark \\ t^2 y_2'' + t y_2' - y_2 &= t^2 \cdot 2t^{-3} + t \cdot (-t^{-2}) - t^{-1} = 0 & \checkmark \end{aligned}$$

Since the function  $y_1(t)/y_2(t)=t^2$  is not constant,  $y_1=y_1(t)$  and  $y_2=y_2(t)$  are linearly independent.

(b+c) The general solution of the inhomogeneous equation has the form  $y(t) = y_h(t) + y_p(t)$ . Since  $y_1=y_1(t)$  and  $y_2=y_2(t)$  are linearly independent solutions of the associated homogeneous ODE, the general solution of the homogeneous ODE is given by

$$y(t) = C_1 t + C_2 t^{-1}.$$

We next use variation of parameters to find a particular solution  $y_p = y_p(t)$  to the inhomogeneous equation. In other words, we would like to find  $v_1=v_1(t)$  and  $v_2=v_2(t)$  such that the function

$$y_p(t) = y_1(t)v_1(t) + y_2(t)v_2(t) = t v_1(t) + t^{-1} v_2(t)$$

solves the ODE. We compute  $y_p'(t)$ , writing the resulting expression symmetrically with respect to  $v_1$  and  $v_2$ :

$$y_p'(t) = (t v_1'(t) + t^{-1} v_2'(t)) + (v_1(t) - t^{-2} v_2(t)).$$

Since the function  $y_p$  involves two parameters,  $v_1$  and  $v_2$ , but has to satisfy only one equation, we can impose one more condition on  $v_1$  and  $v_2$ :

$$t v_1'(t) + t^{-1} v_2'(t) = 0 \implies y_p'(t) = v_1(t) - t^{-2} v_2(t) \implies y_p''(t) = (v_1'(t) - t^{-2} v_2'(t)) + 2t^{-3} v_2(t).$$

Plugging these expressions into the inhomogeneous ODE, we get

$$t^2 (v_1' - t^{-2} v_2' + 2t^{-3} v_2) + t (v_1 - t^{-2} v_2) - (t v_1 + t^{-1} v_2) = 8t^2 e^{2t} \implies t^2 (v_1' - t^{-2} v_2') = 8t^2 e^{2t}.$$

Combining this condition on  $v_1$  and  $v_2$  with the one imposed above, we get:

$$\begin{aligned} \begin{cases} v_1' + t^{-2} v_2' = 0 \\ v_1' - t^{-2} v_2' = 8e^{2t} \end{cases} &\implies \begin{cases} v_1' = 4e^{2t} \\ v_2' = -4t^2 e^{2t} \end{cases} \implies v_1 = 2e^{2t} \\ v_2 &= -4 \int t^2 e^{2t} dt = -2(t^2 e^{2t} - \int 2t e^{2t} dt) = -2t^2 e^{2t} + 2(t e^{2t} - \int e^{2t} dt) = -2t^2 e^{2t} + 2t e^{2t} - e^{2t} \end{aligned}$$

Note that we need to find only one pair  $(v_1, v_2)$  that solves the first system. Putting everything together, we conclude that

$$y_p = y_1 v_1 + y_2 v_2 = 2e^{2t} - t^{-1} e^{2t} \implies \boxed{y(t) = y_h(t) + y_p(t) = C_1 t + C_2 t^{-1} + 2e^{2t} - t^{-1} e^{2t}}$$

### Problem 5 (30pts)

A tank contains  $V_0$  gallons of salt solution of concentration  $\rho_0$  pounds of salt per gallon. Another salt solution containing  $\rho_1$  pounds of salt per gallon is poured into the tank from the top at the constant rate of  $2r$  gallons per minute. A drain is opened at the bottom of the tank allowing the solution to exit at the constant rate of  $r$  gallons per minute, so that the amount of solution in the tank increases. Assume that the solution in the tank is kept perfectly mixed at all times. Let  $\rho(t)$  be the salt concentration in the tank  $t$  minutes after the second solution started pouring into the tank.

(a; 15pts) Show that the concentration  $\rho(t)$  is the solution to the initial value problem

$$\rho'(t) = \frac{2r}{V_0+rt}(\rho_1 - \rho(t)), \quad \rho(0) = \rho_0.$$

(b; 15pts) Determine the number  $T$  of minutes it will take for the solution in the tank to reach the concentration of  $(\rho_0 + \rho_1)/2$ .

(a) By our assumptions,  $\rho = \rho(t)$  satisfies the initial condition. Thus, it remains to determine the rate of change of  $\rho$ . If  $S(t)$  is the amount of salt in the tank at time  $t$ ,

$$S'(t) = \text{rate}_{\text{salt}}^{\text{in}}(t) - \text{rate}_{\text{salt}}^{\text{out}}(t) = \rho^{\text{in}}(t) \cdot \text{rate}_{\text{mix}}^{\text{in}}(t) - \rho^{\text{out}}(t) \cdot \text{rate}_{\text{mix}}^{\text{out}}(t) = \rho_1 \cdot 2r - \rho(t) \cdot r.$$

Since  $S(t) = \rho(t) \cdot V(t)$  and  $V(t) = V_0 + rt$ , it follows that

$$\rho'(t) \cdot (V_0 + rt) + \rho(t) \cdot r = S'(t) = 2r\rho_1 - r\rho(t) \quad \implies \quad \rho'(t) = \frac{2r}{V_0 + rt}(\rho_1 - \rho(t)).$$

(b) We first find the solution  $\rho = \rho(t)$  to the initial value problem in (a). The ODE is separable (as well as linear). We write  $\rho' = d\rho/dt$  and separate variables:

$$\begin{aligned} \frac{d\rho}{dt} = \frac{2r}{V_0+rt}(\rho_1 - \rho) &\iff \frac{d\rho}{\rho_1 - \rho} = \frac{2r dt}{V_0 + rt} \iff \int_{\rho_0}^{\rho} \frac{dy}{\rho_1 - y} = \int_0^t \frac{2r ds}{V_0 + rs} \\ &\iff -(\ln|\rho_1 - \rho| - \ln|\rho_1 - \rho_0|) = 2(\ln|V_0 + rt| - \ln|V_0|) \\ &\iff \ln\left|\frac{\rho_1 - \rho_0}{\rho_1 - \rho}\right| = 2\ln\left(\frac{V_0 + rt}{V_0}\right) \iff \left|\frac{\rho_1 - \rho_0}{\rho_1 - \rho}\right| = \left(1 + \frac{r}{V_0}t\right)^2 \end{aligned}$$

The last relation defines  $\rho = \rho(t)$  implicitly. We use it to find  $T$  such that  $\rho(T) = (\rho_0 + \rho_1)/2$ :

$$\left|\frac{\rho_1 - \rho_0}{\rho_1 - \rho(T)}\right| = \left(1 + \frac{r}{V_0}T\right)^2 \iff 2 = \left(1 + \frac{r}{V_0}T\right)^2 \iff \boxed{T = (\sqrt{2} - 1)V_0/r}$$

*Note 1:* Above we use  $\rho_1 \neq \rho_0$ ; otherwise  $T = 0$ .

*Note 2:* We could instead find the general solution and then solve for the constant to obtain the solution to the IVP in (a) and finally for  $T$ .

### Problem 6 (30pts)

(a; 18pts) Find the general solution to the ODE

$$\mathbf{y}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \mathbf{y}(t).$$

(b; 12pts) Sketch the phase-plane portrait for this ODE, showing all important qualitative information. Determine whether the origin is an asymptotically stable, stable, or unstable equilibrium and why.

(a) The characteristic polynomial in this case is

$$\lambda^2 - (\text{tr} A)\lambda + (\det A) = \lambda^2 - 2\lambda = \lambda(\lambda - 2).$$

Thus, the eigenvalues are  $\lambda_1 = 0$  and  $\lambda_2 = 2$ . We next find corresponding eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ :

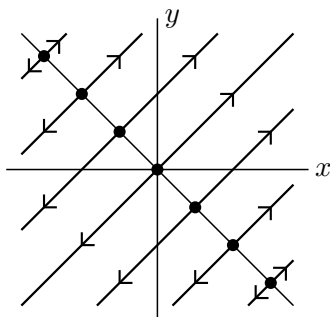
$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} c_1 + c_2 = 0 \\ c_1 + c_2 = 0 \end{cases} \iff c_1 = -c_2 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & 1-\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{cases} \iff c_1 = c_2 \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, the general solution is given by

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C_1, C_2 \in \mathbb{R}$$

(b) The corresponding phase-plane portrait is shown below. The solutions are described by various choices of the constants  $C_1$  and  $C_2$ . As always, the solution “curve” corresponding to  $C_1 = C_2 = 0$  is the origin. In this particular case, if  $C_2 = 0$ , the corresponding solution “curve” is a single point on the line  $y = -x$ , and every point on this line is an equilibrium point. If  $C_2 \neq 0$ , the corresponding solution curve is the ray for  $\mathbf{v}_2$  that starts at  $(C_1 \ -C_1)^t$ .



all points on the line  $y = -x$   
are equilibrium pts

The origin is an unstable equilibrium point, since some solution curves move away from it. This conclusion also follows from the fact that one of the eigenvalues is positive.

### Problem 7 (25pts)

Find the general solution to the ODE

$$\mathbf{y}' = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \mathbf{y}(t).$$

Determine whether the origin is an asymptotically stable, stable, or unstable equilibrium and why.

The general solution is given by

$$\mathbf{y}(t) = e^{tA} \mathbf{v} = C_1 e^{tA} \mathbf{v}_1 + C_2 e^{tA} \mathbf{v}_2 + C_3 e^{tA} \mathbf{v}_3 \quad \mathbf{v} \in \mathbb{R}^3, \quad C_1, C_2, C_3 \in \mathbb{R},$$

where  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is a basis for  $\mathbb{R}^3$ . Since this matrix is upper-triangular, the eigenvalues are  $\lambda_1 = \lambda_2 = \lambda_3 = 2$ .

Since  $A$  has only one eigenvalue, we can compute  $e^{tA}$ :

$$\begin{aligned} tA &= \begin{pmatrix} 2t & 0 & 0 \\ 0 & 2t & 0 \\ 0 & 0 & 2t \end{pmatrix} + \begin{pmatrix} 0 & 3t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2tI + tB; \\ e^{2tI} &= e^{2tI}, \quad e^{tB} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ (2tI)(tB) &= (tB)(2tI) \implies e^{tA} = e^{2tI+tB} = e^{2tI} e^{tB} = e^{2t} \begin{pmatrix} 1 & 3t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \implies \boxed{\mathbf{y}(t) = e^{2t} \begin{pmatrix} 1 & 3t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{v} = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3t \\ 1 \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \end{aligned}$$

Alternatively, we can start by finding a basis for the eigenspace:

$$\begin{pmatrix} 2-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 3c_2 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We next pick a third element for a basis, e.g.  $\mathbf{v}_2 = (0 \ 1 \ 0)^t$ , and find  $\mathbf{v}$  in the eigenspace such that

$$\begin{aligned} A\mathbf{v}_3 &= \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \mathbf{v} + \lambda \mathbf{v}_3 \implies \mathbf{v} = 3\mathbf{v}_1 \implies e^{tA} \mathbf{v}_3 = te^{\lambda t} \mathbf{v} + e^{t\lambda} \mathbf{v}_3 = 3te^{2t} \mathbf{v}_1 + e^{2t} \mathbf{v}_3 \\ \implies \boxed{\mathbf{y}(t) = C_1 e^{2t} \mathbf{v}_1 + C_2 e^{2t} \mathbf{v}_2 + C_3 e^{2t} (3t\mathbf{v}_1 + \mathbf{v}_3) = (C_1 + 3tC_3) e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \end{aligned}$$

Since one of the eigenvalues is positive, there are solutions that move away from the origin. Thus, the origin is an unstable equilibrium point.

### Problem 8 (30pts)

(a; 10pts) Show that the change of variables from Cartesian coordinates  $(x, y) = (x(t), y(t))$  to polar coordinates  $(r, \theta) = (r(t), \theta(t))$  reduces the system of ODEs

$$\begin{cases} x' = y + x(\sqrt{x^2+y^2}-1)(\sqrt{x^2+y^2}-2)^2(\sqrt{x^2+y^2}-4) \\ y' = -x + y(\sqrt{x^2+y^2}-1)(\sqrt{x^2+y^2}-2)^2(\sqrt{x^2+y^2}-4) \end{cases} \quad \text{to} \quad \begin{cases} r' = r(r-1)(r-2)^2(r-4) \\ \theta' = -1 \end{cases}$$

(b; 20pts) Find all equilibrium points and limit cycles for the first system of ODEs in (a) and determine their type. Sketch the corresponding phase-plane portrait in the  $xy$ -plane.

(a) Differentiating both sides of  $r^2 = x^2 + y^2$  and  $\tan \theta = (y/x)$  with respect to  $t$ , we obtain

$$\begin{aligned} 2rr' &= 2xx' + 2yy' = 2x(y + x(r-1)(r-2)^2(r-4)) + 2y(-x + y(r-1)(r-2)^2(r-4)) \\ &= 2r^2(r-1)(r-2)^2(r-4) \quad \implies \quad r' = r(r-1)(r-2)^2(r-4) \\ \frac{\theta'}{\cos^2 \theta} &= \frac{y'}{x} - \frac{yx'}{x^2} = \frac{1}{x^2}(x(-x + y(r-1)(r-2)^2(r-4)) - y(y + x(r-1)(r-2)^2(r-4))) \\ &= \frac{-r^2}{x^2} = \frac{-1}{\cos^2 \theta} \quad \implies \quad \theta' = -1 \end{aligned}$$

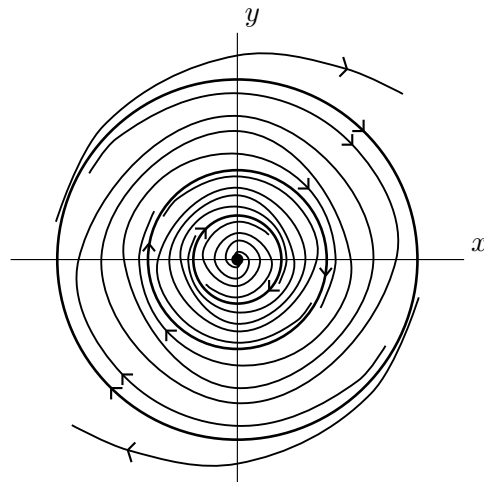
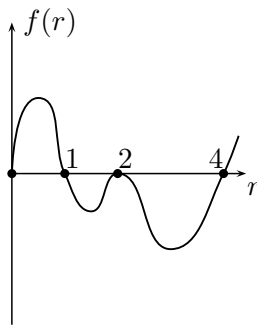
(b) By the second system,  $\theta(t)$  always changes at a constant rate, while  $r$  changes unless  $r=0, 1, 2, 4$ . Thus, the only equilibrium point is the origin  $(x, y) = (0, 0)$ , and the only cycles are the circles of radii 1, 2, and 4 centered at the origin. Sketching the function  $f(r) = r(r-1)(r-2)^2(r-4)$  with  $r \geq 0$ , we find for which values of  $r$  the derivative of  $r = r(t)$  is positive. The  $r$ -phase line is shown below. From this, or from the phase-plane sketch, we see that

$r=1$  is an attracting cycle;  $r=4$  repelling;  $r=2$  is neither

Since  $\theta$  is always decreasing, while  $r$  increases nearly the origin,

$(x, y) = (0, 0)$  is a spiral source, with clockwise rotation

The phase-plane portrait is sketched by combining the  $r$ -phase line with the fact that  $\theta$  is decreasing at a constant rate.



radii of circles: 1, 2, 4  
all spirals/circles rotate clockwise



### Problem 9 (50pts)

(a; 30pts) Find all equilibrium points for the system of ODEs

$$\begin{cases} x' = 2(x + y) \\ y' = (x - 1)(y - 1) \end{cases} \quad x = x(t), \quad y = y(t).$$

Determine the type of each equilibrium point in detail (i.e. stability, spiral/nodal sink/source, saddle, and, if relevant, direction of rotation, slopes).

(b; 20pts) Sketch the phase-plane portrait for the system of ODEs in part (a). You may assume that the phase-plane portrait contains no limit cycles.

(c; Bonus 5pts) Explain why the system of ODEs in (a) has no limit cycles.

(a) The equilibrium points are the solutions  $(x, y)$  of the system:

$$\begin{cases} x' = 2(x + y) = 0 \\ y' = (x - 1)(y - 1) \end{cases} \implies \begin{cases} y = -x \\ x = 1 \text{ or } y = 1 \end{cases}$$

Thus, the equilibrium points are  $\boxed{(1, -1) \text{ and } (-1, 1)}$  The jacobian in this case is:

$$J(x, y) = \begin{pmatrix} 2 & 2 \\ y-1 & x-1 \end{pmatrix} \implies J(1, -1) = \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix} \implies \lambda^2 - 2\lambda + 4 = 0 \implies \lambda_1, \lambda_2 = 1 \pm i\sqrt{3}$$

Thus,  $\boxed{(1, -1) \text{ is a spiral source; direction of rotation=clockwise}}$  Similarly,

$$J(-1, 1) = \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix} \implies \lambda_1 = 2, \lambda_2 = -2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We next find an eigenvector  $\mathbf{v}_2$  for  $\lambda_2$ :

$$\lambda_2 = -2: \begin{pmatrix} 2 - \lambda_2 & 2 \\ 0 & -2 - \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 4c_1 + 2c_2 = 0 \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus,  $\boxed{(-1, -2) \text{ is a saddle point; slope-in}=-2; \text{ slope-out}=0}$

(b) We first indicate the two equilibrium points,  $(1, -1)$  and  $(-1, 1)$ , with large dots. The next step is to sketch the nullclines. The  $x$ -nullcline is described by the equation  $x' = 0$ ; it consists of the line  $y = -x$ . The  $y$ -nullcline is described by the equation  $y' = 0$ ; it consists of the lines  $x = 1$  and  $y = 1$ . The two equilibrium points are the intersections of the  $x$ -nullcline with the two lines making up the  $y$ -nullcline.

Since the direction of rotation around  $(1, -1)$  is clockwise, the flow direction in the middle region to the right of the line  $x = 1$  is down and to the right. We indicate this by labeling the region with  $(+, -)$ . Every time, we cross the  $x$ -nullcline, the  $x$ -sign changes; every time, we cross the  $y$ -nullcline, the  $y$ -sign changes. In this way, we label all the regions, cut out by the nullclines, with  $(\pm, \pm)$  on the first sketch below. In particular, the flow stays on the line  $y = 1$ , pointing away from  $(-1, 1)$ . Thus, this line splits into solution curves, and these solution curves correspond to the two outgoing

solution curves at the saddle point  $(-1, 1)$ . All of this information can also be obtained by looking at the sign of  $x'$  on each segment of the  $y$ -nullcline and at the sign of  $y'$  on each segment of the  $x$ -nullcline.

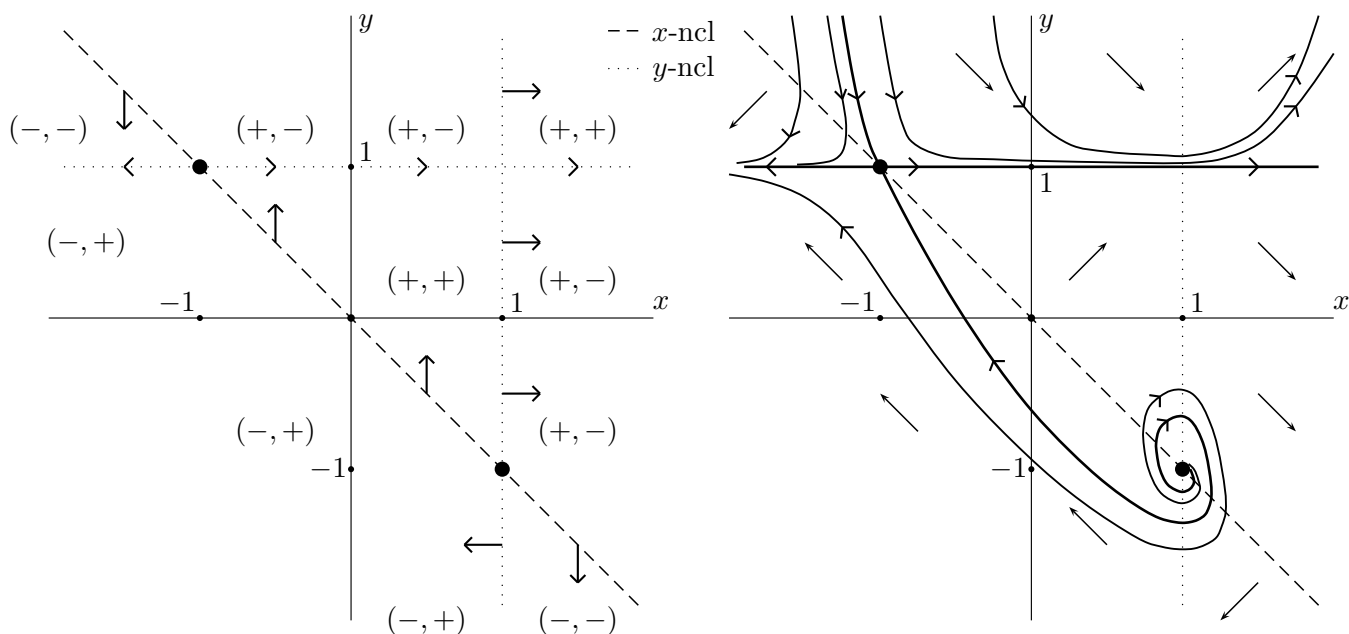
We translate the  $(\pm, \pm)$  labels into arrows on the second sketch. These arrows indicate the general direction of the flow. We also show that the line  $y=1$  is made up of solution curves.

We next sketch the pair of incoming solution curves at the saddle point  $(-1, 1)$ . By (a), their slope at  $(-1, 1)$  is  $-2$ . Thus, one of these curves must come from the middle region above the line  $y=1$  and one from the bottom left region. The first curve, traced backwards and thus against the flow, ascends to the left forever. It never crosses the  $x$ -nullcline  $y=-x$ . In fact, it moves farther and farther away from it and becomes closer and closer to being vertical, even though it will cross every vertical line  $x=a$ , with  $a < -1$ . The reason its slope approaches infinity is that its slope

$$y'/x' = \frac{(x-1)(y-1)}{2(x+y)}$$

increases as  $x$  and  $y$  increase in the absolute value.

The second incoming curve, traced backwards, must descend to the right. It must eventually cross the  $y$ -nullcline  $x=1$ . One way to see this is to notice that if  $x$  is between  $-1$  and  $1$ , then  $x'$  and  $y'$  are both linear in  $y$  and thus increase at a comparable rate. In particular,  $y$  cannot drop to  $-\infty$ , while  $x$  stays between  $-1$  and  $1$ . Once this curve crosses the line  $x=1$ , it will rise to the right and start spiraling around the spiral source  $(1, -1)$ . A priori it may not spiral all the way down to the source, but may instead approach a limit cycle going around the source. However, we are assuming no limit cycle exists. In other words, the lower incoming curve for the saddle point  $(-1, 1)$  originates at the spiral source  $(1, -1)$ .



We now sketch additional solution curves in the various regions of the plane. All solution curves below the horizontal line  $y=1$  originate at the spiral source  $(1, -1)$ . None of these curves can descend forever

to the right; this can be seen by looking at the slope  $y'/x'$  as  $x$  and  $y$  become large. Eventually all of them, with the exception of the lower incoming curve for the saddle point  $(-1, 1)$ , end up below the final ascending piece of this special curve. They then rise to the left and eventually become asymptotic to the line  $y = 1$ . All these curves cross the vertical line  $x = 1$  horizontally and the line  $y = -x$  vertically.

All curves above the line  $y = 1$  and to the left of the upper incoming curve for the saddle point  $(-1, 1)$  come from above the line  $y = -x$ . Their slope approaches infinity when they are traced backwards. They begin by descending steeply to the right to the line  $y = -x$ , which they cross vertically. After that they descend to the left and become asymptotic to the line  $y = 1$ .

All curves above the line  $y = 1$  and to the right of the upper incoming curve for the saddle point  $(-1, 1)$  come from the left of the line  $x = 1$ . Their slope approaches infinity when they are traced backwards. They begin by descending steeply, at first, to the right toward the line  $y = 1$ , and eventually cross the line  $x = 1$  horizontally. After that they ascend to the right and become more and more vertical.

(c) If  $R$  is a simply connected region in the plane (i.e.  $R$  has no holes) and  $f_x + g_y$  does not change sign in  $R$ , then the system of ODEs

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases} \quad x = x(t), \quad y = y(t),$$

has no cycles that are contained entirely in  $R$ . In this case,

$$f_x + g_y = (2(x+y))_x + ((x-1)(y-1))_y = 2 + (x-1) = x+1.$$

This expression is nonnegative for  $x \geq -1$ . Thus, the half-plane  $x \geq -1$  (i.e. everything to the right of the vertical line  $x = -1$ ) contains no limit cycle.

On the other hand, any point to the left of the line  $x = -1$  will flow to the left forever. Thus, no limit cycle can enter the half-plane  $x < -1$ . Combined with the conclusion of the previous paragraph, this implies that our system of ODEs has no limit cycles.

**Problem 10 (30pts)**

Let  $y=y(t)$  be the solution to the initial value problem

$$y' = t/y, \quad y(0) = 1.$$

(a; 15pts) Use the first-order Euler's numerical method with four steps to estimate  $y(2)$ .

(b; 15pts) Use the second-order Runge-Kutta numerical method with two steps to estimate  $y(2)$ .

(a) The step size is  $h=(2-0)/4 = \frac{1}{2}$ , and the first-order method gives

$t_0 = 0$	$y_0 = 1$	$s_0 = t_0/y_0 = 0$	$s_0 h = 0$
$t_1 = \frac{1}{2}$	$y_1 = y_0 + s_0 h = 1$	$s_1 = t_1/y_1 = \frac{1}{2}$	$s_1 h = \frac{1}{4}$
$t_2 = 1$	$y_2 = y_1 + s_1 h = \frac{5}{4}$	$s_2 = t_2/y_2 = \frac{4}{5}$	$s_2 h = \frac{2}{5}$
$t_3 = \frac{3}{2}$	$y_3 = y_2 + s_2 h = \frac{33}{20}$	$s_3 = t_3/y_3 = \frac{10}{11}$	$s_3 h = \frac{5}{11}$
$t_4 = 2$	$y_4 = y_3 + s_3 h = \frac{463}{220}$		

Thus, the resulting estimate for  $y(2)$  is  $\boxed{463/220}$

(b) In this case  $h=(2-0)/2 = 1$ , but we need to find two slopes at each step and average them:

$t_0 = 0$	$y_0 = 1$	$s_{0,1} = t_0/y_0 = 0$	$s_{0,1} h = 0$		
		$s_{0,2} = t_1/(y_0 + s_{0,1} h) = 1$	$s_0 = \frac{s_{0,1} + s_{0,2}}{2} = \frac{1}{2}$	$s_0 h = \frac{1}{2}$	
$t_1 = 1$	$y_1 = y_0 + s_0 h = \frac{3}{2}$	$s_{1,1} = t_1/y_1 = \frac{2}{3}$	$s_{1,1} h = \frac{2}{3}$		
		$s_{1,2} = t_2/(y_1 + s_{1,1} h) = \frac{12}{13}$	$s_1 = \frac{s_{1,1} + s_{1,2}}{2} = \frac{31}{39}$	$s_1 h = \frac{31}{39}$	
$t_2 = 2$	$y_2 = y_1 + s_1 h = \frac{179}{78}$				

Thus, the resulting estimate for  $y(2)$  is  $\boxed{179/78}$