

Math 53 – Spring 2005
Final Exam, June 3, 2005

Instructions: Please answer each question carefully and completely. Make sure you show all your work. No calculators, books or notes allowed. Good luck!

Name _____

Section leader and time _____

1. (10) _____
2. (20) _____
3. (25) _____
4. (15) _____
5. (10) _____
6. (20) _____
7. (10) _____
8. (20) _____
9. (25) _____
10. (15) _____
- Total (170) _____

1. (10 pts.) Find the general solution to the inhomogeneous linear equation

$$y''' - 3y' + 2y = e^{2t}$$

by whatever method you wish (except leaning over and looking at your neighbour's exam).

2. (20 pts.) Sketch the phase portrait in the (t, y) plane for the autonomous equation $y' = y \cos y$. Indicate the equilibria and determine which equilibria are stable. Sketch the solutions $y(t)$ with $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$.

3. Let $P(t)$ be the population (in thousands) of toads in a certain lake near Hamburg in Germany; the time t is measured in months. Suppose that this population function satisfies a logistic equation, $P' = kP(A - P)$, where A is a constant.

a) (15 pts.) Suppose that the stable population is 4,000 toads. In addition, suppose that $P(0) = 2,000$ and $P(10) = 3,000$, find the value of the constant k . (Your answer will involve logarithms, etc.)

3 (cont.) c) (5 pts.) If you've been following the news recently, you will have read that an unknown disease detected in this very lake is causing toads to explode (I am not making this up - go google it after the exam!). Suppose that this disease takes effect at $t = 10$, and that it causes the toad population to decrease at a rate of 5% per month. What is the new ODE satisfied by the toad population for $t \geq 10$?

d) (5 pts.) Is there a stable toad population after the onset of this disease? If so, what is it? Is this a stable equilibrium?

4. (5 pts. each) Find general solutions to the following equations:

(a) $y' + ty + e^{-t^2/2} = 0$

(b) $y' + \sec y \cos t = 0$

4 (cont.) (c) $(2y e^x + \cos y) y' + y^2 e^x - \sin x = 0$

5. (10 pts.) Compute the Laplace transform of the solution $y(t)$ to the initial value problem

$$y'' + 4y' + 4y = \cos 2t \quad y(0) = 1, \quad y'(0) = -2.$$

6. Consider the initial value problem

$$x'' + 9x = g(t), \quad x(0) = x'(0) = 0.$$

(a) (10 pts.) Find the solution $x(t)$ expressed in terms of a convolution of the fundamental solution (i.e. unit impulse response) of this equation and the forcing term $g(t)$. (Recall that the convolution of two functions $f(t)$ and $k(t)$ is defined by $(f * k)(t) = \int_0^t f(t-u)k(u)du$.)

6 (cont.) (b) (10 pts.) Assume $|g(t)| \leq e^t$ for every $t \geq 0$. Using the formula in part (a), it is possible to conclude that the solution $x(t)$ satisfies $|x(t)| \leq \frac{1}{3}(e^t - 1)$ for every $t \geq 0$. Explain this.

7. (10 pts.) Explain why there is no first-order differential equation $x' = f(x, t)$ (where $f(x, t)$ is a smooth function of two variables) for which $x(t) = \sin t$ and $x(t) = \cos t$ are both solutions.

8. (20 pts.) Let k be any real number, and consider the system

$$y' = \begin{bmatrix} 3k & 4 \\ -1 & 5k \end{bmatrix} y.$$

Describe the different types of behaviour (stable node, saddle, etc.) that occur as k takes on different values, and draw rough sketches of the phase portraits for each (in other words, you do not need to find the precise eigenspaces, but do try to give a good indication of what each picture should look like). If the case of repeated roots arises, it is OK if you choose to not sketch the phase portraits in those cases (although we will be very happy if you include the sketches even then too!).

9. Consider the system

$$\begin{aligned}y_1' &= -y_1 + y_2 \\y_2' &= -y_2 + y_3 \\y_3' &= -y_3.\end{aligned}$$

(a) (10 pts.) Find the general solution of this system ‘directly’, i.e. by solving the last equation for y_3 first, inserting this solution into the equation for y_2 to obtain an inhomogeneous equation, solving that and inserting its solution into the equation for y_1 and solving that.

9 (cont.) (b) (7 pts.) Now define the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Calculate e^{At} . (Hint: first calculate $(A + I)^2, (A + I)^3$, etc.)

c) (8 pts.) Finally, use this matrix exponential e^{At} found in part b) to give an alternate derivation of the general solution of the system in part a).

10. (a) (5 pts.) Show that $y_1(t) = t$ is a solution to

$$y'' + \frac{2}{t}y' - \frac{2}{t^2}y = 0.$$

(b) (10 pts.) Another, linearly independent, solution $y_2(t)$ to this same equation can be found by writing $y_2(t) = v(t)t$. Find the ordinary differential equation which is solved by $v(t)$ and then solve it. Why do you know that $y_1(t)$ and $y_2(t)$ are linearly independent?

Scratch paper

More scratch paper