Solutions to Math 41 Final Exam — December 8, 2014

1. (10 points) Find all values of α and β such that

$$Y(t) = At^{\beta}e^{-t}$$

for some constant A is a solution of

$$y'' + \alpha y' + y = e^{-t}.$$

Solution 1. We vary α and see how β must change accordingly. Note that the characteristic equation is $\lambda^2 + \alpha\lambda + 1 = 0$. Thus, if $\lambda = -1$ is to be a root, then we must have $\alpha = 2$. In this case, $\lambda = -1$ is a repeated root, so the solutions to the homogeneous equation take the form e^{-t} and te^{-t}. Therefore, by the method of undetermined coefficients, we should guess a particular solution with $\beta = 2$:

$$\begin{aligned} & Y(t) &=& \quad At^2e^{-t} \\ \Longrightarrow \quad & Y'(t) &=& -At^2e^{-t} + 2Ate^{-t} \\ \Longrightarrow \quad & Y''(t) &=& \quad At^2e^{-t} - 4Ate^{-t} + 2Ae^{-t} \end{aligned}$$

To verify that such a solution actually exists, we compute

$$Y'' + 2Y' + Y = 2Ae^{-t} \implies A = \frac{1}{2}.$$

On the other hand, if $\alpha \neq 2$, then $\lambda \neq -1$ and the homogeneous solutions do not take the form $t^{\beta}e^{-t}$ for any β. We should then guess a particular solution with $\beta = 0$: $Y(t) = Ae^{-t}$. Plugging in, we find

$$Y'' + \alpha Y' + Y = (2-\alpha)Ae^{-t} \implies A = \frac{1}{2-\alpha}.$$

Thus, we must have $\beta = 2$ if $\alpha = 2$ and $\beta = 0$ if $\alpha \neq 2$.

Solution 2. We vary β and see which values of α we need. Compute directly:

$$\begin{aligned} & Y(t) &=& \quad At^{\beta}e^{-t} \\ \Longrightarrow \quad & Y'(t) &=& -At^{\beta}e^{-t} + \quad A\beta t^{\beta-1}e^{-t} \\ \Longrightarrow \quad & Y''(t) &=& \quad At^{\beta}e^{-t} - 2A\beta t^{\beta-1}e^{-t} + A\beta(\beta-1)t^{\beta-2}e^{-t} \end{aligned}$$

$$\implies e^{-t} = Y'' + \alpha Y' + Y = A\beta(\beta-1)t^{\beta-2}e^{-t} + A(\alpha-2)\beta t^{\beta-1}e^{-t} + A(2-\alpha)t^{\beta}e^{-t}.$$

Matching terms, we find that we need $\beta = 0$, 1, or 2. If $\beta = 0$, then we have

$$A(2-\alpha) = 1 \implies A = \frac{1}{2-\alpha}, \quad \alpha \neq 2.$$

If $\beta = 1$, then

$$A(\alpha-2) + A(2-\alpha)t = 1 \implies \left\{ \begin{array}{l} A(\alpha-2) = 1 \\ A(\alpha-2) = 0 \end{array} \right\} \implies \text{no solution.}$$

If $\beta = 2$, then

$$2A + 2A(\alpha-2)t + A(2-\alpha)t^2 = 1 \implies \left\{ \begin{array}{rl} 2A & = 1 \\ 2A(\alpha-2) & = 0 \\ A(\alpha-2) & = 0 \end{array} \right\} \implies A = \frac{1}{2}, \quad \alpha = 2.$$

Thus, we must have $\alpha \neq 2$ if $\beta = 0$ and $\alpha = 2$ if $\beta = 2$.

2. Consider the linear differential equation

$$ty'' + ty' - y = 0, \quad t > 0.$$

(a) (2 points) Show that $y(t) = t$ is a solution.

We simply compute that if $y = t$, then

$$ty'' + ty' - y = t \cdot 0 + t \cdot 1 - t = t - t = 0.$$

(b) (4 points) Abel's theorem states that the Wronskian of two solutions of the second-order equation

$$y'' + p(t)y' + q(t)y = 0$$

is given by

$$W(t) = c \exp\left[-\int p(t)\,dt\right]$$

for some constant c. Use this to compute the Wronskian for the equation above.

We rewrite the equation as

$$y'' + y' - \frac{1}{t}y = 0.$$

Then we must have $p(t) = 1$, so $\int p(t)\,dt = t$ (any antiderivative will work) and

$$W(t) = c \exp\left[-\int p(t)\,dt\right] = ce^{-t}.$$

We provide a derivation of Abel's theorem for completeness. From the definition, if y_1, y_2 are two solutions to the differential equation, then we have

$$W = y_1 y_2' - y_1' y_2,$$

so by the product rule,

$$W' = y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2 = y_1 y_2'' - y_1'' y_2.$$

Using the fact that y_1, y_2 satisfy $y'' = -p(t)y' - q(t)y$, we get

$$W' = y_1(-py_2' - qy_2) - (-py_1' - qy_1)y_2 = -py_1y_2' + py_1'y_2 = -pW.$$

In other words, $W(t)$ satisfies $W'(t) = -p(t)\,W(t)$; solving this gives the formula in Abel's theorem.

(c) (6 points) Use the Wronskian from (b) to determine a second linearly independent solution. You may leave all integrals unsimplified. *Hint*: The Wronskian of two solutions $y_1(t)$ and $y_2(t)$ is

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

Solution 1. We compute from the formula with $y_1 = t$ that

$$ce^{-t} = W(t) = \begin{bmatrix} t & y_2 \\ 1 & y_2' \end{bmatrix} = ty_2' - y_2.$$

For linear independence, we need $c \neq 0$; choose $c = 1$. Then we have a linear equation in y_2:

$$y_2' - \frac{1}{t}y_2 = \frac{e^{-t}}{t}.$$

Computing the integrating factor,

$$\mu(t) = \exp\left(-\int \frac{dt}{t}\right) = \exp(-\ln t) = \frac{1}{t},$$

we have

$$\left(\frac{1}{t}y_2\right)' = \frac{1}{t}y_2' - \frac{1}{t^2}y_2 = \frac{e^{-t}}{t^2} \implies y_2 = t\int \frac{e^{-t}}{t^2}\,dt.$$

Solution 2. Alternatively, we could finish from $e^{-t} = ty_2' - y_2$ by noting that $ty_2' - y_2 = -ty_2''$ since y_2 satisfies the given differential equation, giving that

$$y_2'' = -\frac{e^{-t}}{t} \implies y_2 = -\int^t \left(\int^s \frac{e^{-u}}{u}\,du\right) ds.$$

We can verify that both of these formulas are equivalent and do indeed work:

$$y_2' = \int \frac{e^{-t}}{t^2}\,dt + \frac{e^{-t}}{t} \qquad \implies \qquad y_2'' = \frac{e^{-t}}{t^2} - \frac{e^{-t}}{t^2} - \frac{e^{-t}}{t} = -\frac{e^{-t}}{t}$$

as expected, so

$$ty_2'' + ty_2' - y_2 = -e^{-t} + t\int \frac{e^{-t}}{t^2}\,dt + e^{-t} - t\int \frac{e^{-t}}{t^2}\,dt = 0,$$

as desired.

3. (9 points) Suppose that a differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

can be made exact through multiplying by an integrating factor $\mu(x,y) = \mu(z)$ for some differentiable function $z = f(x,y)$. Prove that μ satisfies

$$\mu'(z) = \left(\frac{M_y - N_x}{Nf_x - Mf_y}\right)\mu(z).$$

In order for $\mu(z)$ to be an integrating factor, it has to satisfy

$$\begin{aligned}(\mu(z)\,M(x,y))_y &= (\mu(z)\,N(x,y))_x\\ \mu(z)_y\,M(x,y) + \mu(z)\,M_y(x,y) &= \mu(z)_x\,N(x,y) + \mu(z)\,N_x(x,y)\end{aligned}$$

Note that by the chain rule, we can find $\mu(z)_x$ and $\mu(z)_y$:

$$\begin{aligned}\mu(z)_x &= \frac{\partial\mu(z)}{\partial z}\frac{\partial z}{\partial x} = \mu'(z)f_x\\ \mu(z)_y &= \frac{\partial\mu(z)}{\partial z}\frac{\partial z}{\partial y} = \mu'(z)f_y\end{aligned}$$

By substitution, we obtain

$$\begin{aligned}\mu'(z)\,f_yM + \mu(z)\,M_y &= \mu'(z)\,f_xN + \mu(z)\,N_x\\ \mu'(z)(Nf_x - Mf_y) &= \mu(z)(M_y - N_x)\\ \mu'(z) &= \left(\frac{M_y - N_x}{Nf_x - Mf_y}\right)\mu(z)\end{aligned}$$

4. (a) (5 points) Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} t, & 0 < t \leq 1, \\ e^{-t}, & 1 < t. \end{cases}$$

By definition,

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 te^{-st}\,dt + \int_1^\infty e^{-(s+1)t}\,dt.$$

The first integral can be computed by parts:

$$\int_0^1 te^{-st}\,dt = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\bigg|_0^1 = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}, \quad s > 0,$$

while the second integral is just

$$\int_1^\infty e^{-(s+1)t}\,dt = -\frac{e^{-(s+1)t}}{s+1}\bigg|_1^\infty = \frac{e^{-(s+1)}}{s+1}, \quad s > -1.$$

Therefore,

$$\mathcal{L}\{f(t)\}(s) = -\frac{e^{-s}}{s} + \frac{e^{-(s+1)}}{s+1} + \frac{1}{s^2} - \frac{e^{-s}}{s^2}, \quad 0 \neq s > -1.$$

(b) (5 points) Show that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t.$$

Let $f(t) = \sin t$. Then

$$\mathcal{L}\left\{\frac{1}{2} t \sin t\right\}(s) = \frac{1}{2}\mathcal{L}\{tf(t)\}(s) = -\frac{1}{2}\mathcal{L}\{f(t)\}'(s) = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{s}{(s^2+1)^2}.$$

(c) (5 points) Calculate the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+1)^2}\right\}.$$

You may use the result from (b) without justification.

Let $f(t) = (1/2)t \sin t$. Then

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0) = \frac{s^2}{(s^2+1)^2},$$

so the inverse Laplace transform is just

$$f'(t) = \frac{1}{2}\frac{d}{dt}(t \sin t) = \frac{1}{2}(t \cos t + \sin t).$$

5. (12 points) Solve the initial value problem

$$y^{(4)} - 16y = 0, \qquad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

Solution 1. Find the characteristic equation by substituting $y = e^{\lambda t}$:

$$0 = \lambda^4 - 16 = (\lambda^2 + 4)(\lambda^2 - 4) = (\lambda + 2)(\lambda - 2)(\lambda^2 + 4),$$

which has roots $\lambda = \pm 2, \pm 2i$. Hence, the general solution is

$$\begin{aligned} & y(t) &&= Ae^{2t} + Be^{-2t} + C\cos(2t) + D\sin(2t) \\ \implies & y'(t) &&= 2Ae^{2t} - 2Be^{-2t} - 2C\sin(2t) + 2D\cos(2t) \\ \implies & y''(t) &&= 4Ae^{2t} + 4Be^{-2t} - 4C\cos(2t) - 4D\sin(2t) \\ \implies & y'''(t) &&= 8Ae^{2t} - 8Be^{-2t} + 8C\sin(2t) - 8D\cos(2t) \end{aligned}$$

for constants A, B, C, D. It remains to match the initial conditions:

$$\left.\begin{aligned} 1 &= y(0) = A + B + C \\ 1 &= y'(0) = 2A - 2B + 2D \\ -2 &= y''(0) = 4A + 4B - 4C \\ 0 &= y'''(0) = 8A - 8B - 8D \end{aligned}\right\} \implies A = \frac{1}{4}, \quad B = 0, \quad C = \frac{3}{4}, \quad D = \frac{1}{4}.$$

Note. Strictly speaking, we introduced this technique only for second-order equations, but the principle is clearly the same. We can also come to the same conclusion using only second-order methods by writing

$$0 = \left(\frac{d^4}{dt^4} - 16\right) y = \left(\frac{d^2}{dt^2} + 4\right) \underbrace{\left(\frac{d^2}{dt^2} - 4\right) y}_{z} = 0 \implies \begin{cases} z'' + 4z = 0 \\ y'' - 4y = z \end{cases}$$

and solving for $z(t)$ then for $y(t)$.

Solution 2. Let $Y(s) = \mathcal{L}\{y\}(s)$ and take the Laplace transform:

$$\begin{aligned} (s^4 - 16)Y(s) &= s^3 y(0) + s^2 y'(0) + s y''(0) + y'''(0) \\ (s^2 + 4)(s^2 - 4)Y(s) &= s^3 + s^2 - 2s \\ (s + 2)(s - 2)(s^2 + 4)Y(s) &= s(s + 2)(s - 1), \end{aligned}$$

so

$$Y(s) = \frac{s(s-1)}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs + 2C}{s^2 + 4} = \frac{A(s^2+4) + (Bs + 2C)(s-2)}{(s-2)(s^2+4)}$$

for constants A, B, C, i.e.,

$$\left.\begin{aligned} 1 &= A + B \\ -1 &= -2B + 2C \\ 0 &= 4A - 4C \end{aligned}\right\} \implies A = \frac{1}{4}, \quad B = \frac{3}{4}, \quad C = \frac{1}{4}$$

on matching coefficients. Thus,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \frac{1}{4}e^{2t} + \frac{3}{4}\cos(2t) + \frac{1}{4}\sin(2t). \end{aligned}$$

6. Consider the autonomous nonlinear system

$$x' = 2xy,$$
$$y' = e^x - y^2.$$

(a) (3 points) Find all critical points (equilibrium solutions).

At equilibrium, $x' = y' = 0$. To set $x' = 0$, we need either $x = 0$ or $y = 0$. If $x = 0$, then $0 = y' = 1 - y^2$ gives $y = \pm 1$, so $(0, \pm 1)$ are equilibria. On the other hand, if $y = 0$, then we require $0 = y' = e^x$, which has no solution. Therefore, $(0, \pm 1)$ are the only equilibria.

(b) (3 points) Determine the type and stability of each critical point.

Compute the Jacobian:

$$J(x, y) = \begin{bmatrix} 2y & 2x \\ e^x & -2y \end{bmatrix} \quad \Longrightarrow \quad J(0, 1) = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}, \quad J(0, -1) = \begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix}.$$

The Jacobians at both equilibria have eigenvalues $\lambda_{1,2} = \pm 2$ and so are (unstable) saddle points.

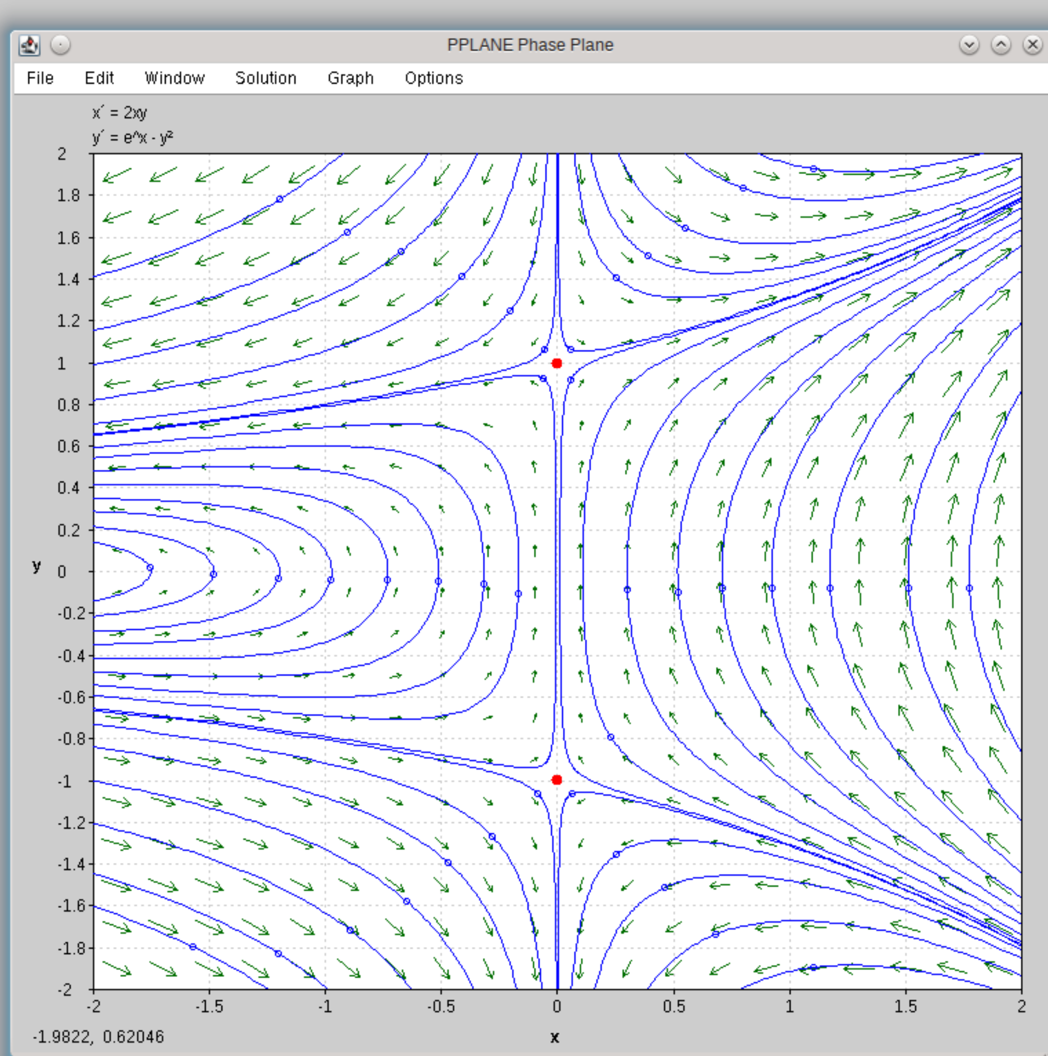
(c) (4 points) Draw a phase portrait around each critical point.

Compute eigenvalue/eigenvector pairs:

$$(0, +1): \qquad \lambda_1 = 2, \quad v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(0, -1): \qquad \lambda_1 = 2, \quad v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \quad v_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Use this information to guide nearby trajectories: around $(0, 1)$, solutions converge to the equilibrium along $(0, 1)^\mathsf{T}$ and diverge along $(4, 1)^\mathsf{T}$; similarly, around $(0, -1)$, solutions converge along $(4, -1)^\mathsf{T}$ and diverge along $(0, 1)^\mathsf{T}$. A combined phase portrait is given on the next page.



(d) (5 points) Find the equations of all solution curves.

Reduce the autonomous system to a first-order equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^x - y^2}{2xy} \quad \Longrightarrow \quad (y^2 - e^x)\,dx + 2xy\,dy = 0,$$

which is exact and has the solution

$$H(x,y) = xy^2 - e^x = \text{const.}$$

7. (15 points) Find the general solution of the inhomogeneous linear differential equation

$$y'' + y = \sec t.$$

For this problem, we can only apply variation of parameters. To do this, we need to find a fundamental set of solutions to the equation

$$y'' + y = 0.$$

By solving the characteristic equation $\lambda^2 + 1 = 0$, we get $\lambda = \pm i$, and hence a fundamental set of solutions is

$$y_1(t) = \cos t, \quad y_2(t) = \sin t.$$

One can also compute the Wronskian of $\{y_1, y_2\}$ by

$$W[y_1, y_2](t) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = 1.$$

Then a particular solution is given by

$$\begin{aligned} y_p(t) &= y_1(t) \int \frac{-y_2(t)\sec t}{W[y_1, y_2](t)} dt + y_2(t) \int \frac{y_1(t)\sec t}{W[y_1, y_2](t)} dt \\ &= \cos t \int (-\sin t) \frac{1}{\cos t} dt + \sin t \int \cos t \frac{1}{\cos t} dt \\ &= \cos t \ln|\cos t| + t \sin t. \end{aligned}$$

(If this looks unfamiliar, write down the corresponding first-order system in $(y, y')^{\mathsf{T}}$ and work through the variation of parameters formula in that setting.) The general solution is then

$$y(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t) = \cos t \ln|\cos t| + t \sin t + c_1 \cos t + c_2 \sin t$$

for constants c_1 and c_2.

8. Consider the nonlinear initial value problem

$$\frac{t}{2}y' + y - \sqrt{y} = 0, \quad y(1) = 0.$$

(a) (2 points) Find a solution by inspection.

Clearly $y \equiv 0$ is a solution.

(b) (6 points) Use the change of variables $y(t) = [u(t)]^2$ to find a second solution.

By plugging in $y = u^2$, we have

$$\frac{t}{2} \cdot 2uu' + u^2 - \sqrt{u^2} = 0, \quad \text{i.e.,} \quad tuu' + u^2 - |u| = 0.$$

So our new equation is

$$\begin{cases} tuu' + u^2 - u = 0, & u \geq 0 \\ tuu' + u^2 + u = 0, & u < 0 \end{cases} \implies \begin{cases} tu' + u - 1 = 0, & u \geq 0 \\ tu' + u + 1 = 0, & u < 0 \end{cases}$$

since we want to find some solution other than $u \equiv 0$. Notice that if $u(t)$ is a solution to the first equation above, then $-u(t)$ is a solution to the second one; furthermore, both give the same result for $y(t) = [u(t)]^2$. Thus, we only need to solve, say,

$$tu' + u - 1 = 0, \quad u \geq 0,$$

which has the solution

$$u(t) = 1 - \frac{1}{t}, \qquad t < 0, \quad t \geq 1$$

on using the initial condition $u(1) = \sqrt{y(1)} = 0$. In order to get a second solution for $y(t)$, we can therefore take

$$y(t) = \begin{cases} 0, & t < 1, \\ (1 - 1/t)^2, & t \geq 1. \end{cases}$$

This is a differentiable function on all of $\mathbb{R}$ satisfying the original initial value problem since both parts are solutions.

(c) (4 points) Why does this not contradict the theorem on the existence of a unique solution?

This does not contradict the uniqueness theorem, since if we denote

$$f(t, y) = \frac{2}{t}(\sqrt{y} - y),$$

then the partial derivative f_y is given by

$$f_y = \frac{1}{t}\left(\frac{1}{\sqrt{y}} - 2\right),$$

which is not continuous around $(t, y) = (1, 0)$.