Stanford University Department of Mathematics

Math 53, Fall 2007 — Final Exam

Instructor : Samuel Lisi

Date: 10 December 2007
Duration: 3 hours

FAMILY NAME : ______________________________

GIVEN NAME(S) : ______________________________

STUDENT NUMBER : ______________________________

YOUR SIGNATURE: ______________________________

DO NOT OPEN THIS TEST UNTIL INSTRUCTED TO DO SO.

INSTRUCTIONS:

- Your signature above indicates that you have abided by the Stanford Honor Code while writing this test.
- There are eight questions. The last part of question 8 is a bonus.
- You may quote theorems from your textbook or from class if you make an appropriate reference.
- Show all your work.
- No electronic devices of any kind (e.g. calculators, cell-phones) are allowed.
- There is a table of Laplace transform identities on the last page of the exam. You may use these identities without proof, unless the question indicates otherwise.

Question	Marks
1 [10 pts]	
2 [10 pts]	
3 [15 pts]	
4 [15 pts]	
5 [15 pts]	
6 [15 pts]	
7 [15 pts]	
8 [20 pts]	
Total [100 points]	

1. (a) i. Find a solution to the initial value problem [10 pts]

$$y' = ty^2, \quad y(0) = 1.$$

ii. What is the interval of existence of this solution?

iii. Is this solution unique? If yes, explain why. If no, provide a second solution.

(b) Find the general solution to the differential equation

$$y' + 2ty = \mathrm{e}^{-t^2} \sin(t).$$

[10 pts]

2. Find the integral curve of

$$\alpha = (\sin(x+y) + 2x)dx + (y^2 + \sin(x+y))dy$$

that passes through the point $(\pi, 0)$. (You may define this curve implicitly.)

[15 pts]

3. (a) Solve the initial value problem :

$$y'' + 4y = 2\cos(2t), \quad y(0) = 0, y'(0) = 0.$$

(b) Solve the initial value problem :

$$y'' + 4y = 2\cos(2t), \quad y(0) = 1, y'(0) = -1.$$

[15 pts]

4. Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

   (a) Find the matrix exponential e^{At}.

   (b) Solve the initial value problem :

$$\mathbf{y}' = A\mathbf{y} + \begin{pmatrix} \mathrm{e}^{2t}\sin(t) \\ \mathrm{e}^{2t} \\ 0 \end{pmatrix}, \qquad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

[15 pts]

5. (a) Sketch the phase portrait for $\mathbf{y}' = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \mathbf{y}$.

(b) Determine whether $\mathbf{0}$ is asymptotically stable, stable or unstable.

[15 pts]

6. Consider the nonlinear system of differential equations given by :

$$x' = 2x(2 - x)$$
$$y' = y(1 + x^2)$$

   (a) Find the equilibrium solutions.

   (b) For each equilibrium solution you found in (a), determine its stability.

[15 pts]

7. (a) Find the Laplace transform of the solution to the initial value problem

$$y'' + 2y' - y = \cos(2t), \qquad y'(0) = 1, y(0) = -1.$$

(You do **not** need to solve for $y(t)$.)

(b) Find the inverse Laplace transform of $\dfrac{-4}{(s-1)^2(s-3)}$.

The two parts of this question are unrelated.

8. Consider the nonlinear system of differential equations given by : [a–c: 10 pts]

$$x' = 2y$$
$$y' = -2x - 4x^3$$

(a) Show that $(0,0)$ is the only equilibrium solution.

(b) What does the linearization at $(0,0)$ tell us about stability?

(c) Suppose that $(x(t), y(t))$ is a solution to the nonlinear system. Show that it is an integral curve of

$$\alpha = (2x + 4x^3)dx + 2ydy.$$

(d) [Bonus – 10 pts] Use the fact that α is exact to conclude that $(0,0)$ is stable, but not asymptotically stable.

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Laplace transform identities

In the following, let f be piecewise continuous of exponential order, and $F = \mathcal{L}(f)$:

$$
\begin{aligned}
\mathcal{L}(\mathrm{e}^{ct} f(t)) &= F(s-c) \\
\mathcal{L}(tf(t)) &= -F'(s) \\
\mathcal{L}(1) &= \frac{1}{s} \qquad \text{for } s > 0 \\
\mathcal{L}(t^n) &= \frac{n!}{s^{n+1}} \qquad \text{for } s > 0 \\
\mathcal{L}(\sin(at)) &= \frac{a}{s^2+a^2} \qquad \text{for } s > 0 \\
\mathcal{L}(\cos(at)) &= \frac{s}{s^2+a^2} \qquad \text{for } s > 0 \\
\mathcal{L}(\mathrm{e}^{at}) &= \frac{1}{s-a} \qquad \text{for } s > a \\
\mathcal{L}(\mathrm{e}^{at}\sin(bt)) &= \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a \\
\mathcal{L}(\mathrm{e}^{at}\cos(bt)) &= \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a \\
\mathcal{L}(t^n\,\mathrm{e}^{at}) &= \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a
\end{aligned}
$$

If f is piecewise differentiable, and both f and f' are of exponential order, then there exists $a > 0$ so that :

$$\mathcal{L}(f'(t)) = sF(s) - f(0) \qquad \text{for } s > a.$$