Synchrosqueezed wave packet transform for 2D mode decomposition

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Main Question How to analyze a typical time signal which can be viewed as a superposition of several simple components?

Application

In many applications,for examples, engeneering, medicine, and finance, time signals are decomposed into several simple components to characterize the structure, to make prediction and to identify determinant components.

Character

1. Time signals are often nonlinear, generated from dynamical systems obeying nonlinear equations
2. Time signals would be non-stationary in the sense that there may be jumps or changes with important significance

Time-Frequency Analysis

Time-frequency analysis study a signal in both the time and frequency domains simultaneously via a certain time-frequency transform.

- Linear methods: efficient, easy for reconstruction, but poor resolution
- Quadratic methods: better resolution, but higher computational cost, more difficult to reconstruct and non-physical interference between multiple components

Synchrosqueezing for better Time-Frequency Analysis

To address the resolution limitation of linear time-frequency analysis, Daubechies et al propose the synchrosqueezing technique for a sharpened time-frequency representation, the synchrosqueezed wavelet transforms.

Empirical mode decomposition

- The EMD method proposed and improved by Huang et al uses a sifting process to break down a signal into a summation of intrinsic mode functions (IMF), which induce stability problems in the presence of noise.
- A new method named EEMD is aimed to deal with these issues, but is difficult to analyze mathematically.

Data-driven mode decomposition via compressive sensing

Inspired by the EMD, Hou et al propose a data-driven mode decomposition method via compressive sensing.

- Adaptive data-driven method without predetermined basis functions
- Sparsest mode decomposition via proper optimization

Motivation

- Mode decomposition methods for higher dimensional time signals have important applications in engineering. However, useful tools for higher dimensional mode decomposition are still under development.
- Higher dimensional EEMD, which applies 1D EEMD on each dimension and adopts a certain well defined combination technique afterward to get a higher dimensional decomposition, is proposed by Huang et al recently.
- Higher dimensional mode decomposition method based on time-frequency analysis is still an open problem. In this talk, we introduce one by generalizing the essential idea, the synchrosqueezing technique in Daubechies' paper, and sellecting appropriate time-frequency representation.

Wave packet instead of wavelet

In high dimension, time-frequency representation should be able to distinguish waves with different frequency vectors.

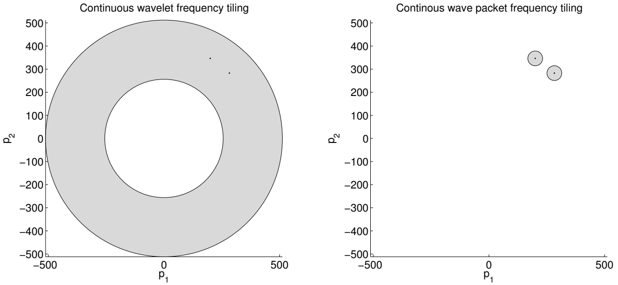


Figure: Consider the superposition of two plane waves $e^{2\pi i p\cdot x}$ and $e^{2\pi i q\cdot x}$ with the same frequency ($|p| = |q|$) but different wave numbers ($p \neq q$).

Definition 1: Wave Packets

Given the mother wave packet $w(x)$ and $s \in (1/2, 1)$, the family of wave packets $\{w_{pb}(x), p, b \in R^2\}$ is defined as

$$w_{pb}(x) = |p|^s w(|p|^s(x-b))e^{2\pi i(x-b)\cdot p},$$

or equivalently in Fourier domain

$$\widehat{w_{pb}}(\xi) = |p|^{-s} e^{-2\pi i b\cdot\xi}\hat{w}(|p|^{-s}(\xi - p)).$$

Definition 2: Wave Packet Transform

The wave packet transform of a function $f(x)$ is a function of $p, b \in R^2$

$$\begin{aligned} W_f(p,b) &= \langle w_{pb}, f\rangle = \int \overline{w_{pb}(x)} f(x)dx \\ &= \langle \widehat{w_{pb}}, \hat{f}\rangle = \int \overline{\widehat{w_{pb}}(\xi)}\hat{f}(\xi)d\xi \end{aligned}$$

As a simple example, let us consider the wave packet transform for a plane wave function

$$f(x) = \alpha e^{2\pi i N\beta\cdot x},$$

where α and β are non-zero constants of order $O(1)$ and N is a sufficiently large constant. The instantaneous wavevector is $N\beta$ and we have

$$\begin{aligned} W_f(p,b) &= \int_{R^2} \alpha e^{2\pi i N\beta\cdot x} |p|^s w(|p|^s(x-b)) e^{-2\pi i (x-b)\cdot p} dx \\ &= |p|^{-s}\alpha \int_{R^2} e^{2\pi i N\beta\cdot(b+|p|^{-s}y)} w(y) e^{-2\pi i p|p|^{-s}y} dy \\ &= |p|^{-s}\alpha e^{2\pi i N\beta\cdot b}\overline{\hat{w}(|p|^{-s}(N\beta-p))}. \end{aligned}$$

Remark: Since $\widehat{w}(\xi)$ is compactly supported in the unit ball, for each fixed b the coefficients $W_f(p,b)$ are non-zero if p satisfies

$$|p - N\beta| \le |p|^s.$$

The derivative of $W_f(p,b)$ with respect to b and $W_f(p,b)$ satisfy the following equation:

$$\frac{\nabla_b W_f(p,b)}{2\pi i W_f(p,b)} = \frac{2\pi i N\beta |p|^{-s}\alpha e^{2\pi i N\beta\cdot b}\overline{\hat{w}(|p|^{-s}(N\beta - p))}}{2\pi i |p|^{-s}\alpha e^{2\pi i N\beta\cdot b}\overline{\hat{w}(|p|^{-s}(N\beta - p))}} = N\beta$$

for $W_f(p,b) \neq 0$.

This motivates us to define the instantaneous wavevector estimation for a general function $f(x)$ as follows.

Definition 3: Instantaneous Wavevector Estimation

The instantaneous wavevector estimation of a function $f(x)$ at (p,b) is

$$v_f(p,b) = \frac{\nabla_b W_f(p,b)}{2\pi i W_f(p,b)}$$

for $p, b \in R^2$ such that $W_f(p,b) \neq 0$.

Given the wavevector estimation $v_f(p,b)$, the synchrosqueezing step reallocates the information in the phase space and provides a sharpened phase space representation of $f(x)$.

Definition 4: Synchrosqueezed Energy Distribution

Given $f(x)$, for $v, b \in R^2$, $W_f(p,b)$, and $v_f(p,b)$, the synchrosqueezed energy distribution $T_f(v,b)$ is

$$T_f(v,b) = \int |W_f(p,b)|^2 \delta(v_f(p,b) - v)dp.$$

Remark:

- If v_f approximate the instantaneous wavevector accurately, the synchrosqueezed energy is nonzeros at v when the signal contains an instantaneous wavevector v.
- Nonzero W_f has a spreading of width $O(|p|^s)$ around the instantaneous wavevector, which is worse in the sense of resolution.

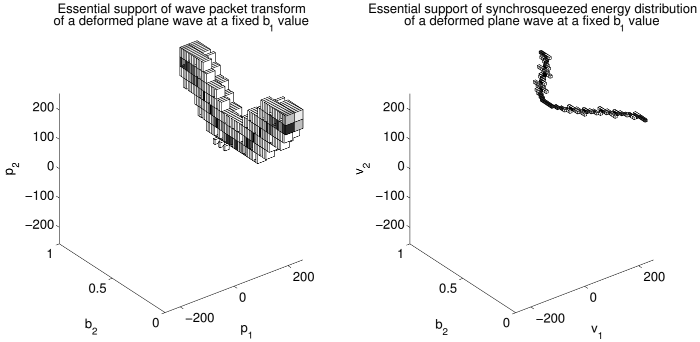


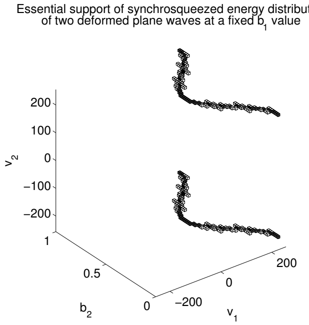
Figure: Synchrosqueezed wave packet transform applied to a deformed plane wave $f(x) = \alpha(x)e^{2\pi iN\phi(x)}$. Left: The essential support of the wave packet transform $W_f(p,b)$ at a fixed b_1 value. Right: The essential support of the synchrosqueezed energy distribution $T_f(v,b)$ at the same b_1 value. The essential support of $W_f(p,b)$ has been reallocated to form a sharp phase space representation $T_f(v,b)$.

An imformal example for mode decomposition

Let

$$f(x) = e^{2\pi i N\phi_1(x)} + e^{2\pi i N\phi_2(x)},$$

with smooth phases $N\phi_1(x)$ and $N\phi_2(x)$ for sufficiently large N. Let us assume that $N\nabla\phi_1(x)$ and $N\nabla\phi_2(x)$ are well-separated from each other.



- The essential support of $T_f(v,b)$ separates into two disjoint regions U_1 and U_2, which can be identified with standard clustering algorithms
- Once U_1 and U_2 are identified, we can extract individual modes with

$$f_1(x) = \int_{v_f(p,b)\in U_1} \tilde{w}_{pb}(x) W_f(p,b)dpdb,$$
$$f_2(x) = \int_{v_f(p,b)\in U_2} \tilde{w}_{pb}(x) W_f(p,b)dpdb$$

where the set of functions $\{\tilde{w}_{pb}(x), p, b \in R^2\}$ is a dual frame of $\{w_{pb}(x), p, b \in R^2\}$.

Analysis of the Transform

Key analysis: How well does the instantaneious wavevector estimation work?

In this subsection we show that, for a superposition of multiple components with well-separated instantaneous wavevectors, the synchrosqueezed wave packet transform is able to estimate these instantaneous wavevectors.

Definition 5: Intrinsic Mode Function

A function $f(x) = \alpha(x)e^{2\pi i N\phi(x)}$ is an intrinsic mode function of type (M, N) if $\alpha(x)$ and $\phi(x)$ satisfy

$$\alpha(x) \in C^\infty, \quad |\nabla\alpha| \leq M, \quad 1/M \leq \alpha \leq M$$
$$\phi(x) \in C^\infty, \quad 1/M \leq |\nabla\phi| \leq M, \quad |\nabla^2\phi| \leq M.$$

Definition 6: Well-Separated Superposition

A function $f(x)$ is a well-separated superposition of type (M, N, K) if

$$f(x) = \sum_{k=1}^{K} f_k(x)$$

where each $f_k(x) = \alpha_k(x) e^{2\pi i N \phi_k(x)}$ is an intrinsic mode function of type (M, N) and they satisfy the separation condition

$$|N\nabla\phi_k(b) - N\nabla\phi_l(b)| \geq 2^{1+s}(|N\nabla\phi_k(b)|^s + |N\nabla\phi_l(b)|^s)$$

for any $1 \leq k, l \leq K$. We denote by $F(M, N, K)$ the set of all such functions.

Theorem

For a function $f(x)$ and $\epsilon > 0$, we define

$$R_{f,\epsilon} = \{(p,b) : |W_f(p,b)| \geq |p|^{-s}\sqrt{\epsilon}\}$$

and

$$Z_{f,k} = \{(p,b) : |p - N\nabla\phi_k(b)| \leq |p|^s\}$$

for $1 \leq k \leq K$. For fixed M and K, there exists a constant $\epsilon_0(M,K) > 0$ such that for any $\epsilon \in (0,\epsilon_0)$ there exists a constant $N_0(M,K,\epsilon) > 0$ such that for any $N > N_0(M,K,\epsilon)$ and $f(x) \in F(M,N,K)$ the following statements hold.

(i) $\{Z_{f,k} : 1 \leq k \leq K\}$ are disjoint and $R_{f,\epsilon} \subset \bigcup_{1\leq k\leq K} Z_{f,k}$;

(ii) For any $(p,b) \in R_{f,\epsilon} \cap Z_{f,k}$,

$$\frac{|v_f(p,b) - N\nabla\phi_k(b)|}{|N\nabla\phi_k(b)|} \lesssim \sqrt{\epsilon}.$$

Outline of the algorithm

For a given superposition $f(x)$ of several well-separated components, the synchrosqueezed wave packet transform consists of the following steps:

1. Apply the wave packet transform to obtain $W_f(p, b)$ and the gradient $\nabla_b W_f(p, b)$;
2. Compute the approximate instantaneous wavevector $v_f(p, b)$ and perform synchrosqueezing to get $T_f(v, b)$;
3. Use a clustering algorithm to identify the support of the new phase space representation $T_f(v, b)$ of different intrinsic mode functions;
4. Reconstruct each intrinsic mode function using the dual frame.

We will describe in detail the discrete synchrosqueezed wave packet transform.

For simplicity, we consider functions that are periodic over the unit square $[0,1)^2$ in 2D. Let

$$X = \{(n_1/L, n_2/L) : 0 \le n_1, n_2, < L, n_1, n_2 \in Z\}$$

be the $L \times L$ spatial grid at which these functions are sampled. The corresponding $L \times L$ Fourier grid is

$$\Xi = \{(\xi_1, \xi_2) : -L/2 \le \xi_1, \xi_2 < L/2, \xi_1, \xi_2 \in Z\}.$$

We simply discretize the position space with an $L_B \times L_B$ uniform grid:

$$B = \{(n_1/L_B, n_2/L_B) : 0 \le n_1, n_2 < L_B, n_1, n_2 \in Z\}.$$

For each fixed $p \in P$ and $b \in B$, the discrete wave packet is defined through its Fourier transform as

$$\widehat{w_{pb}}(\xi) = \frac{1}{L_p} e^{-2\pi i b \cdot \xi} g_p(\xi)$$

for $\xi \in \Xi$, where g_p is a carefully defined function.

For a function $f(x)$ defined on $x \in X$, the discrete wave packet transform is a map from $\ell_2(X)$ to $\ell_2(P \times B)$, defined by

$$W_f(p,b) = \langle w_{pb}, f \rangle = \langle \hat{w}_{pb}, \hat{f} \rangle = \frac{1}{L_p} \sum_{\xi \in \Xi} e^{2\pi i b \cdot \xi} g_p(\xi) \hat{f}(\xi).$$

Algorithm: Fast Forward transform from $f(x)$ to $W_f(p,b)$

1: Compute $\hat{f}(\xi)$ with $\xi \in \Xi$ from $f(x)$ with $x \in X$ using an $L \times L$ forward FFT.
2: **for** each $p \in P$ **do**
3: Form $g_p(\xi)\hat{f}(\xi)$ on the support of $g_p(\xi)$
4: Wrap the result modulo L_B onto the domain $[-L_B/2, L_B/2)^2$
5: Apply an $L_B \times L_B$ inverse FFT to the wrapped result
6: Multiple the result by L_B/L_p to get $W_f(p,b)$ for all $b \in B$
7: **end for**

For a function $h(p,b)$ in $\ell_2(P \times B)$, the transpose of the wave packet transform is given by

$$W_h^t(x) := \sum_{p \in P, b \in B} h(p,b) w_{pb}(x) (L_p/L_B)^2.$$

Algorithm: Fast Transpose Operator from $h(p,b)$ to $W_h^t(x)$

1: **for** each $p \in P$ **do**
2: Multiply $h(p,b)$ for each $b \in B$ by L_p/L_B
3: Apply an $L_B \times L_B$ forward FFT to the product
4: Unwrap the result modulo L_B onto the support of $g_p(\xi)$
5: Multiply the unwrapped data with $g_p(\xi)$ and add the product to get $\hat{f}(\xi)$
6: **end for**
7: Compute $f(x)$ with $x \in X$ from $\hat{f}(\xi)$ with $\xi \in \Xi$ using an $L \times L$ inverse FFT.

We define the discrete gradient $\nabla_b W_f(p,b)$ in a similar way

$$\nabla_b W_f(p,b) = \nabla_b \langle \hat{w}_{pb}, \hat{f} \rangle = \sum_{\xi \in \Xi} \frac{1}{L_p} 2\pi i \xi e^{2\pi i b \cdot \xi} g_p(\xi) \hat{f}(\xi).$$

Algorithm: Discrete Gradient Operator from $f(x)$ to $\nabla_b W_f(p,b)$

1: Compute $\hat{f}(\xi)$ with $\xi \in \Xi$ from $f(x)$ with $x \in X$ using an $L \times L$ forward FFT.
2: **for** each $p \in P$ **do**
3: Form $2\pi i \xi g_p(\xi)\hat{f}(\xi)$ on the support of $g_p(\xi)$
4: Wrap the result modulo L_B onto the domain $[-L_B/2, L_B/2)^2$
5: Apply an $L_B \times L_B$ inverse FFT to each component of the wrapped result
6: Multiple the result by L_B/L_p to get $\nabla_b W_f(p,b)$ for all $b \in B$
7: **end for**

To specify the synchrosqueezed energy distribution $T_f(v, b)$, we first place in the Fourier domain a two dimensional Cartesian grid of stepsize Δ:

$$V = \{(n_1\Delta, n_2\Delta) : n_1, n_2 \in Z\}.$$

At each $v = (n_1\Delta, n_2\Delta) \in V$, we associate a cell D_v centered at v

$$D_v = \left[(n_1 - \frac{1}{2})\Delta, (n_1 + \frac{1}{2})\Delta\right) \times \left[(n_2 - \frac{1}{2})\Delta, (n_2 + \frac{1}{2})\Delta\right).$$

Then the discrete synchrosqueezed energy distribution is defined as

$$T_f(v, b) = \sum_{(p,b):v_f(p,b)\in D_v} |W_f(p, b)|^2 \cdot (L_p/L_B)^2.$$

After synchrosqueezing, $T_f(v, b)$ is essentially supported in the phase space near the K "discrete" surfaces $\{(N\phi_k(b), b), b \in B\}$. The next step is to decompose the essential support of $T_f(v, b)$ into K clusters,

Algorithm: General Spectral Clustering on set $S = s_1, \ldots, s_n$

1: Construct the matrix $A = (\alpha_{ij})_{ij} \in R^{n\times n}$ with distance function $\alpha_{ij} = \exp(-|s_i - s_j|^2/\sigma^2)$ if $i \neq j$, and $\alpha_{ii} = 0$, $\forall i$. Here σ is an input parameter.

2: Let D to be a diagonal matrix such that $D_{ii} = \sum_{j=1}^{n} \alpha_{ij}$ and define the Laplacian-type matrix $L = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$.

3: Choose the K largest orthogonal eigenvectors of L, say $v_1,\ldots,v_K$, and stack them horizontally to get the matrix $V = [v_1, v_2, \ldots, v_K] \in R^{n\times K}$. The entries of V are denoted by v_{ij}.

4: Define the matrix $M = (m_{ij})$ with $m_{ij} = v_{ij}/(\sum_j v_{ij}^2)^{1/2}$, which means normalizing the rows of V.

5: Consider each row of M as a point in R^K and then partition these n points into K clusters with the K-means algorithm.

6: If row i of M is assigned to cluster j, then assign the original point s_i to cluster j.

- In the current setting, we choose a threshold parameter $\eta > 0$, define the set S to be

$$\{(v, b) : v \in V, b \in B, T_f(v, b) \geq \eta\},$$

and apply the above algorithm to S. The resulting clusters are defined to be $U_1, \ldots, U_K$.

- In the final step, we recover each intrinsic mode function by computing.

$$f_k(x) = \sum_{(p,b):v_f(p,b)\in U_k} W_f(p, b) w_{pb}(x)(L_p/L_B)^2.$$

This step can be carried out efficiently by restricting $W_f(p, b)$ to the set $\{(p, b) : v_f(p, b) \in U_k\}$ and applying transpose operator to the restriction for each k.

Instantaneous wavevecter extraction

We test the accuracy of the estimated instantaneous wavevector $v_f(p,b)$. Let $f(x)$ be a deformed plane wave

$$f(x) = \alpha(x) e^{2\pi i N\phi(x)}.$$

The estimate $v_f(p,b)$ should approximates $N\nabla\phi(b)$. We define the mean estimated instantaneous frequency

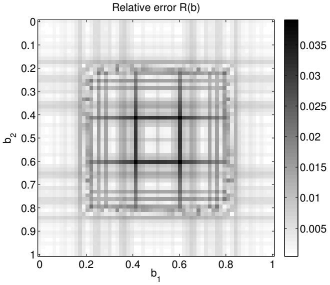
$$v_f^m(b) = \frac{\sum_p |W_f(p,b)|^2 v_f(p,b)}{\sum_p |W_f(p,b)|^2}$$

We can define the relative error $R(b)$ between $v_f^m(b)$ and the exact instantaneous frequency $N\nabla\phi(b)$ as

$$R(b) = \frac{|v_f^m(b) - N\nabla\phi(b)|}{|N\nabla\phi(b)|}.$$

Error of instantaneous wavevector extraction

Example 1. We perform this test on $f(x)$ with $\alpha(x) = 1$, $\phi(x) = \phi(x_1, x_2) = x_1 + x_2 + 0.1\sin(2\pi x_1) + 0.1\sin(2\pi x_2)$, and $N = 135$. $R(b)$ is of order 10^{-2}, which agrees with theorem on that the relative approximation error is of order $O(\sqrt{\epsilon})$.

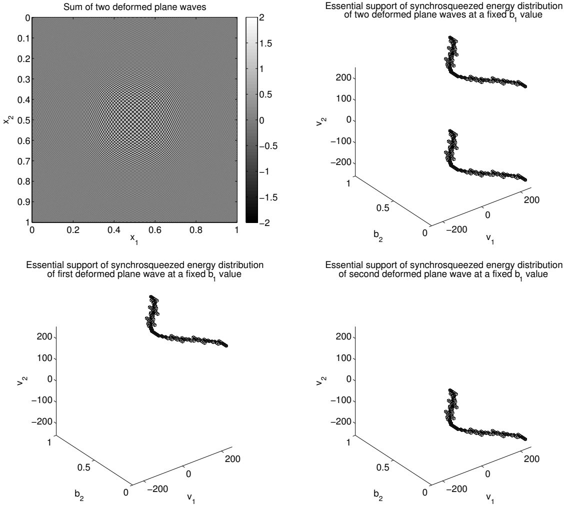


Noiseless Signal of Two Deformed Plane Waves

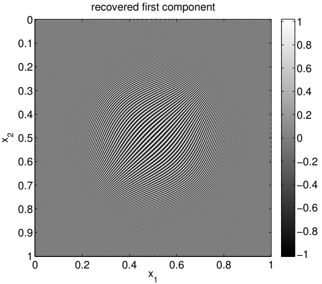
Example 2. Here $f(x)$ is a sum of two deformed plane waves

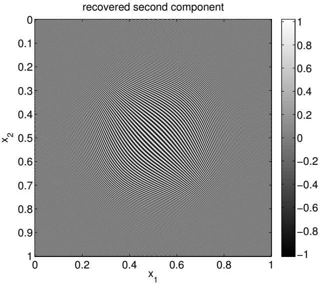
$$f(x) = e^{2\pi i N\phi_1(x)} + e^{2\pi i N\phi_2(x)}$$
$$\phi_1(x) = \phi_1(x_1, x_2) = x_1 + x_2 + \beta \sin(2\pi x_1) + \beta \sin(2\pi x_2)$$
$$\phi_2(x) = \phi_2(x_1, x_2) = -x_1 + x_2 - \beta \sin(2\pi x_1) + \beta \sin(2\pi x_2)$$

with $N = 135$ and $\beta = 0.1$.
We applied synchrosqueezed wave package transform on f and use spectral clustering method to extract each component.


Sum of two deformed plane waves
Essential support of synchrosqueezed energy distribution of two deformed plane waves at a fixed b_1 value
Essential support of synchrosqueezed energy distribution of first deformed plane wave at a fixed b_1 value
Essential support of synchrosqueezed energy distribution of second deformed plane wave at a fixed b_1 value
x_1
x_2
v_1
v_2
b_2

Two Recovered Components



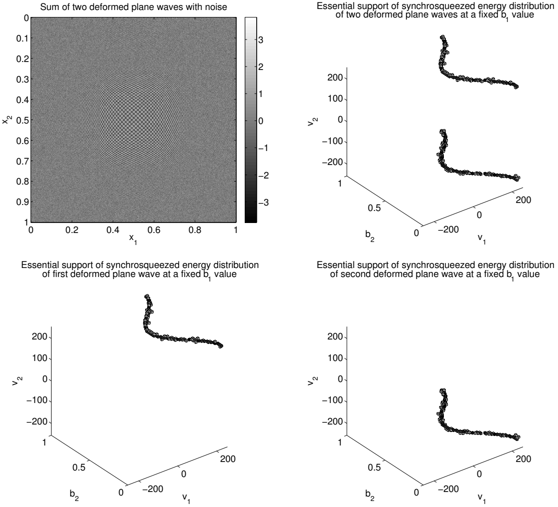


Noicy Signal of Two Deformed Plane Waves

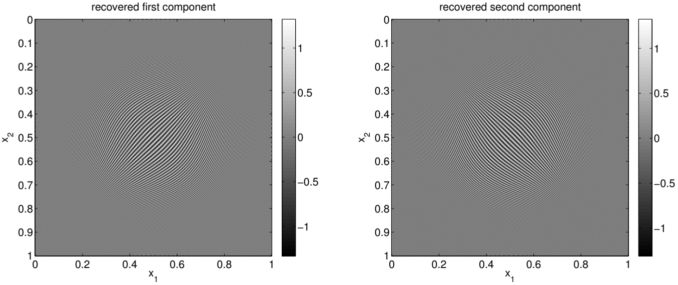
The proposed synchrosqueezed wave packet transform is also rather robust to noise. To demonstrate this, let $f(x)$ be the superposition of two deformed plane waves and a noise term

$$f(x) = e^{2\pi i N\phi_1(x)} + e^{2\pi i N\phi_2(x)} + n(x),$$

where $n(x)$ is an isotropic complex Gaussian random noise with zero mean and variance $\sigma^2 = 0.5$. In order to reduce the influence of noise, we set up a threshold parameter $\delta \approx 3\sigma^2$ and keep only the values of $T_f(v, b)$ that are greater than δ.


Sum of two deformed plane waves with noise
x1
x2
Essential support of synchrosqueezed energy distribution of two deformed plane waves at a fixed b1 value
Essential support of synchrosqueezed energy distribution of first deformed plane wave at a fixed b1 value
Essential support of synchrosqueezed energy distribution of second deformed plane wave at a fixed b1 value
v1
v2
b2

Two Recovered Components from Noicy Data

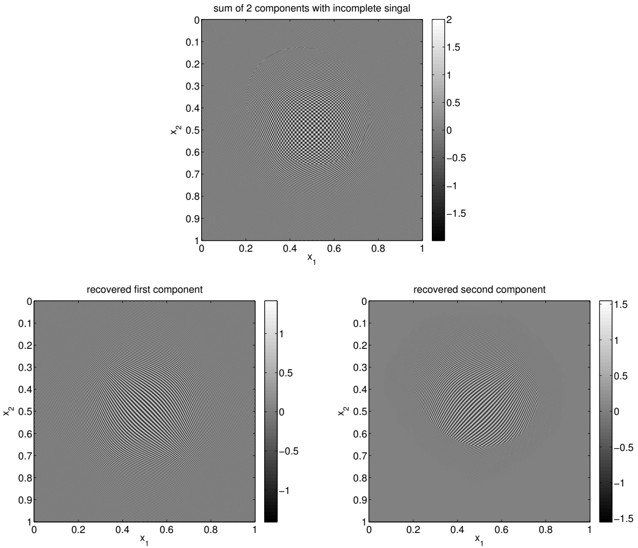


Signal of Incompleted Deformed Plane Waves

We will show that the synchrosqueezed wave packet transform still works quite well under this more general setting. Here we choose $f(x)$ to be the superposition of two components, one of which is incomplete:

$$f(x) = \chi(x) \cdot e^{2\pi i N \phi_1(x)} + e^{2\pi i N \phi_2(x)},$$
$$\phi_1(x) = \phi_1(x_1, x_2) = -(x_1 + \beta \sin(2\pi x_1)) + (x_2 + \beta \sin(2\pi x_2)),$$
$$\phi_2(x) = \phi_2(x_1, x_2) = (x_1 + \beta \sin(2\pi x_1)) - (x_2 + \beta \sin(2\pi x_2)),$$

where $N = 135$, $\beta = 0.1$, and $\chi(x)$ is an indicator function of an ellipse in $[0, 1)^2$.


sum of 2 components with incomplete singal
recovered first component
recovered second component
x1
x2

Conclusion

- This method is an initial step for 2D mode decomposition via time-frequency analysis. Higher dimensional method can be achieved by applying higher dimensional wave package transform.
- A wide range of well-separated nonlinear wave components can be extracted accurately with analytic and numerical proof.
- Robust against noise and working well on non-stationary data.

Future Work

- The synchrosqueezed wave packet transform has a geometric scaling parameter s, which is in $(1/2, 1)$. One natural question is whether it is possible to generalize or modify the synchrosqueezing idea so that it will work for the wave atom case where $s = 1/2$.
- Curvelet is an optimal tool to represent images with curve discontinuity and to identify isolated wavefronts. A synchrosqueezed curvelet transform will have better applications in some problems.
- Robust clustering algorithms are necessary when we generalize the definition of well-separated IMF such that they can have intersecting surfaces after synchrosqueezed wave package transform.