Synchrosqueezed Transforms in Signal and Image Analysis

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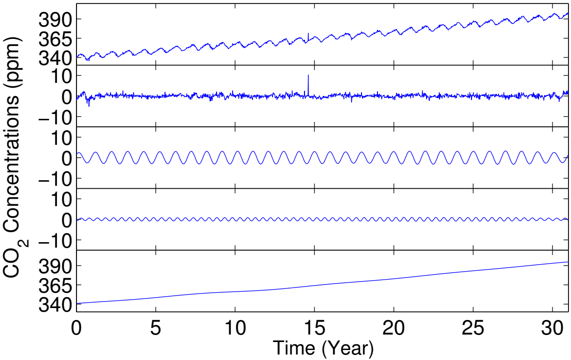
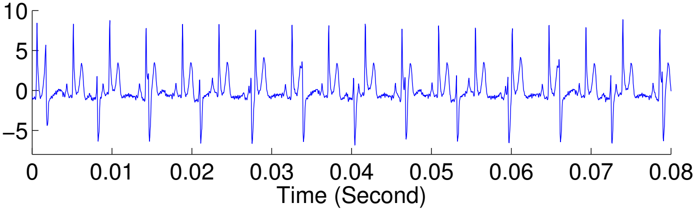
Climate study:

- CO_2 concentration data observed by National Oceanic and Atmospheric Administration at Mauna Loa (MLO).
- General climbing temperature, semiannual cycle, annual cycle, and more?



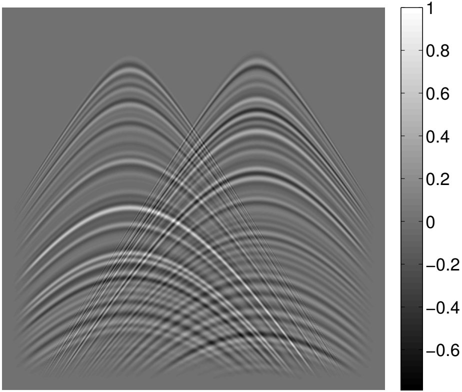
Medical study:

- A supperposition of two ECG signals.
- Healthy people or patients?

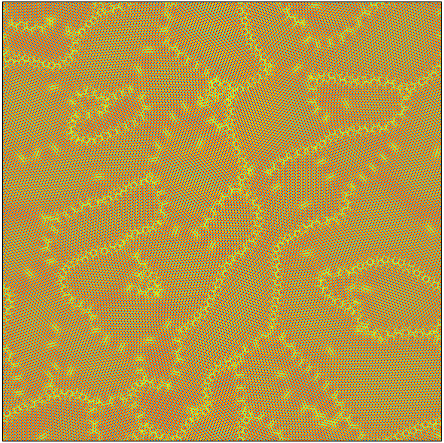


Geophysics:

- A supperposition of several wave fields.
- Can we separate them?



Material science: Crystal segmentations, crystal rotations, crystal defects, crystal deformations.



1D Problem:

Known: A superposition of wave-like components

$$f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) e^{2\pi i N_k \phi_k(t)}. \tag{1}$$

Unknown: Number K, components $f_k(t)$, instantaneous amplitudes $\alpha_k(t)$, instantaneous frequencies $N_k \phi_k(t)$.

General 1D Problem:

Known: A super position of general wave-like components

$$f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) s_k(2\pi N_k \phi_k(t)), \tag{2}$$

Unknown: Some extra unknown information, general wave shape functions $s_k(t)$.

2D Problem:

Known: A superposition of wave-like components

$$f(x) = \sum_{k=1}^{K} f_k(x) = \sum_{k=1}^{K} \alpha_k(x) e^{2\pi i N_k \phi_k(x)}. \quad (3)$$

Unknown: Number K, components $f_k(x)$, instantaneous amplitudes $\alpha_k(x)$, instantaneous frequencies $N_k \phi_k(x)$.

General 2D Problem for mode decomposition:

Known: A super position of general wave-like components

$$f(x) = \sum_{k=1}^{K} f_k(x) = \sum_{k=1}^{K} \alpha_k(x) s_k(2\pi N_k \phi_k(x)), \quad (4)$$

Unknown: Basic unknown information and general wave shape functions $s_k(x)$.

General 2D Problem for image segmentation:

Known: A super position of general wave-like components

$$f(x) = \sum_{k=1}^{K} X_{\Omega_k}(x) f_k(x) = \sum_{k=1}^{K} X_{\Omega_k}(x) \alpha_k(x) s_k(2\pi N_k \phi_k(x)), \qquad (5)$$

Unknown: Basic unknown information, general wave shape functions $s_k(x)$ and non-overlapping regions Ω_k.

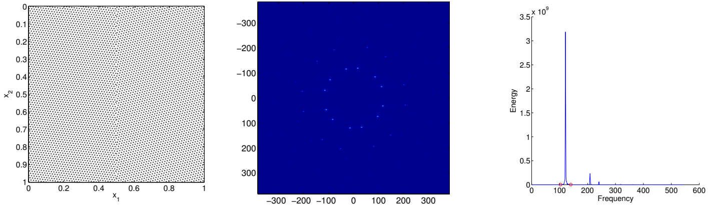


Figure : Left:simulation of a crystal image. Middle: 2D Fourier power spectrum. Right: Radially Fourier power spectrum.

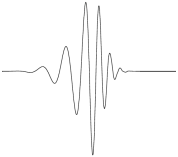
Time-frequency representatin: continuous wavelets

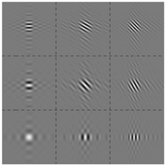
Wavelet transform of a signal s:

$$W_s(a,b) = \langle s(t), \phi_{ab}(t) \rangle = \int s(t) a^{-1/2} \overline{\phi(\frac{t-b}{a})} dt.$$

EX: $s(t) = A\cos(\omega t)$.

$$\begin{aligned} W_s(a,b) &= \frac{1}{2\pi} \int \widehat{s}(\xi) a^{1/2} \overline{\widehat{\phi}(a\xi)} e^{ib\xi} d\xi \\ &= \frac{A}{4\pi} a^{1/2} \overline{\widehat{\phi}(a\omega)} e^{ib\omega}. \end{aligned}$$





Synchrosqueezing for better Time-Frequency Analysis

Resolution limitation of the wavelet transform

Synchrosqueezing wavelet transforms (SWT) (Daubechies et al)

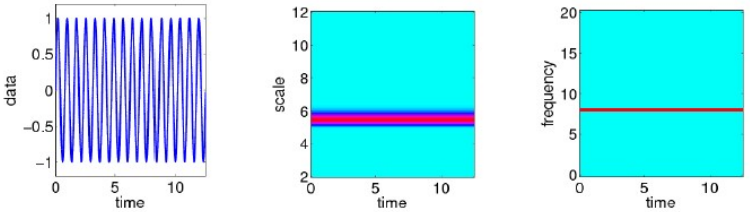


Figure : Numerical example by Daubechies et al, signal $f(t) = sin(8t)$.

Recall: $W_s(a,b) = \frac{A}{4\pi} a^{1/2} \overline{\widehat{\phi}(a\omega)} e^{ib\omega}$.

Definition: Instantaneous frequency

$$\omega_s(a,b) = -i(W_s(a,b))^{-1} \frac{\partial}{\partial b} W_s(a,b) = \omega.$$

Definition: Synchrosqueezed wavelet transform (SWT)

The synchrosqueezed wavelet transform is given by

$$\mathcal{T}_s(\omega,b) = \int_{A(b)} W_s(a,b) a^{-3/2} \delta(\omega_s(a,b) - \omega) da,$$

where $A(b) = \{a : W_s(a,b) \neq 0\}$.

1D Mode decomposition:

$$f(t) = f_1(t) + f_2(t) = \alpha_1(t)e^{2\pi i t^2} + \alpha_2(t)e^{2\pi i t}$$

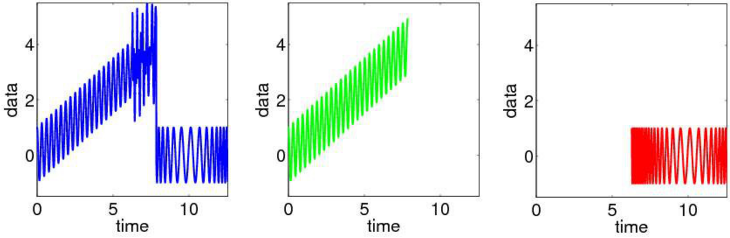


Figure : A superposition of two modes. Courtesy of [Daubechies-Lu-Wu 11].

General intrinsic mode function (Hau-Tieng Wu):

$$f(t) = s_1(2\pi N\phi(t)) = \sum_n \widehat{s}(n) e^{2\pi i n N\phi(t)}$$

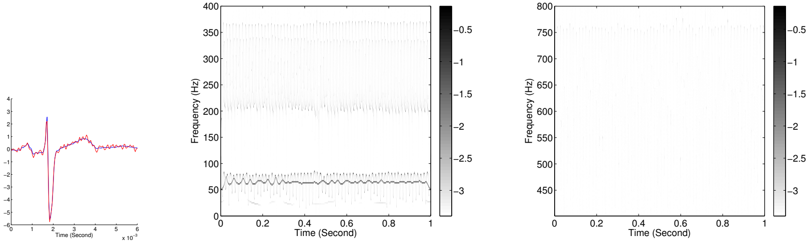


Figure : An ECG signal and its SS wavelet transform.

1D Synchrosqueezed wave packet transform:
Wave packets $\{w_{ab}(t) : |a| \geq 1, b \in R\}$ with $s \in (1/2, 1)$:

$$w_{ab}(t) = |a|^{s/2} w(|a|^s(t-b)) e^{2\pi i(t-b)a},$$

or equivalently, in the Fourier domain as

$$\widehat{w_{ab}}(\xi) = |a|^{-s/2} e^{-2\pi i b\xi} \widehat{w}(|a|^{-s}(\xi - a)).$$

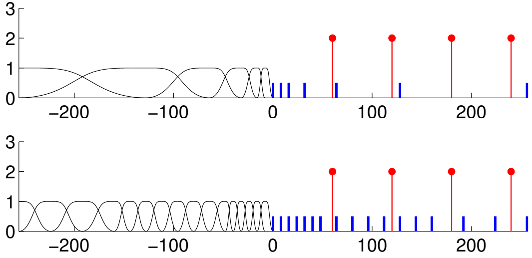


Figure : Comparison of tilings. Top: wavelets. Buttom: wave packets.

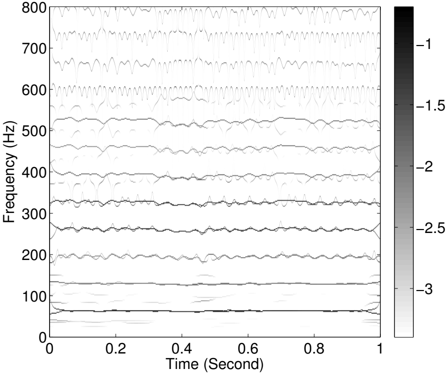


Figure : The synchrosqueezed wave packet transform of an ECG signal.

1D General Mode decomposition:

$$
\begin{aligned}
f(t) &= s_1(2\pi N_1 \phi_1(t)) + s_2(2\pi N_2 \phi_2(t)) \\
&= \sum_n \widehat{s_1}(n) e^{2\pi i n N_1 \phi_1(t)} + \sum_n \widehat{s_2}(n) e^{2\pi i n N_2 \phi_2(t)}
\end{aligned}
$$

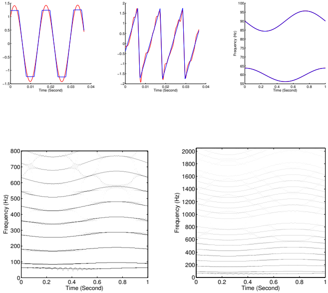


Figure : A superposition of two general intrinsic mode functions

Diffeomorphism based spectral analysis: find n and $|\widehat{s_j}(n)|$ such that $\widehat{s_j}(n) \neq 0$.
Define $p_k(t) = \frac{1}{m_k}\int_0^t n_k N_k \phi_k'(x)dx$. Then

$$\begin{aligned} h_k(t) &= \frac{f \circ p_k^{-1}(t)}{|\widehat{s_k}(n_k)|\alpha_k \circ p_k^{-1}(t)} \\ &= \sum_{n=-\infty}^{\infty} \frac{\widehat{s_k}(n)}{|\widehat{s_k}(n_k)|} e^{2\pi i(\frac{nm_k}{n_k}t + nN_k\phi_k(0))} \\ &\quad + \sum_{j\neq k}\sum_{n=-\infty}^{\infty} \frac{\widehat{s_j}(n)}{|\widehat{s_k}(n_k)|}\frac{\alpha_j \circ p_k^{-1}(t)}{\alpha_k \circ p_k^{-1}(t)} e^{2\pi i n N_j \phi_j \circ p_k^{-1}(t)}. \end{aligned}$$

Find $(\tau, j) = \arg\max_{(\xi,k)} |\widehat{h_k}(\xi)|$. Let $g(t) = e^{2\pi i \tau t}$.

$$\begin{aligned} |\widehat{s_j}(n_j)|\alpha_j(t) g \circ p_j(t) &\approx |\widehat{s_j}(n_j)|\alpha_j(t) e^{2\pi i \frac{nm_j}{n_j} p_j(t)} \\ &= |\widehat{s_j}(n_j)| e^{-2\pi i n N_j \phi_j(0)} \alpha_j(t) e^{2\pi i n N_j \phi_j(t)}. \end{aligned}$$

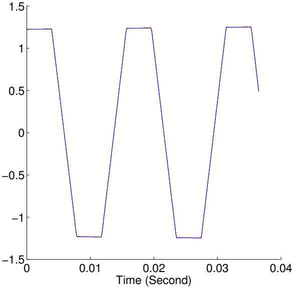


Figure : Blue: Real signals. Red: Reconstructed results. Two recovered general modes provided by the DSA method.

2D mode decomposition

1. A superposition of several wave-like components with diferent oscillatory patterns.
2. Applications in acoustic and electromagnetic scattering, seismic imaging.
3. Differ in wavenumbers: e.g. $e^{i(x_1+x_2)}$ and $e^{i2(x_1+x_2)}$.
4. Differ in directions: e.g. $e^{i(x_1+x_2)}$ and $e^{i(x_1-x_2)}$.

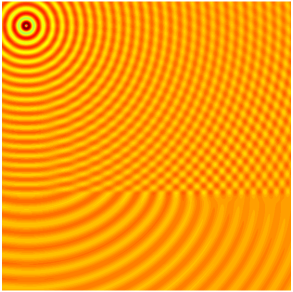
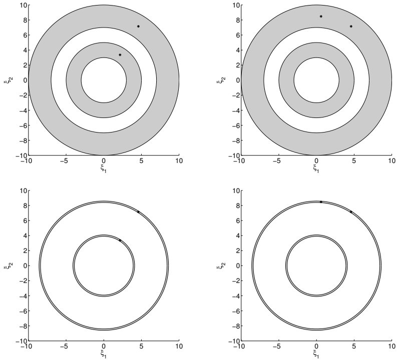


Figure : A solution of Helmoltz equation with a single point source in a layered medium.

2D synchrosqueezed wavelet transform:

Consider two plane waves $e^{ip\cdot x}$ and $e^{iq\cdot x}$ again.



Top: Supports of continuous wavelets and plane waves in the Fourier domain. radial separation.

Bottom: Supports of the synchrosqueezed transform. No angular separation.

Question: How to obtain the angular separation?

Answer: Anisotropic synchrosqueezed wave packet transforms.

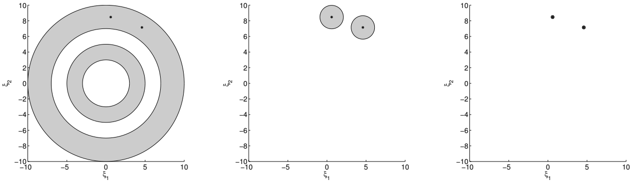


Figure : Left:suports of continuous wavelets. Middle: suports of wave packets. Right: synchrosqueezed wave packet transform.

Definition: Wave Packets

Given the mother wave packet $w(x)$ and $s \in (1/2, 1)$, the family of wave packets $\{w_{pb}(x), p, b \in R^2\}$ is defined as

$$w_{pb}(x) = |p|^s w(|p|^s(x-b))e^{2\pi i(x-b)\cdot p},$$

or equivalently in Fourier domain

$$\widehat{w_{pb}}(\xi) = |p|^{-s} e^{-2\pi i b\cdot\xi}\hat{w}(|p|^{-s}(\xi - p)).$$

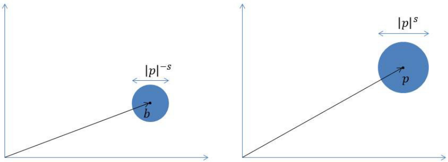


Figure : Left:the support of $w_{pb}(x)$ in spacial domain. Right: the support of $\widehat{w_{pb}}(\xi)$ in the Fourier domain.

Definition: 2D Wave Packet Transform

The wave packet transform of a function $f(x)$ is a function of $p, b \in R^2$

$$W_f(p,b) = \int \overline{w_{pb}(x)} f(x)dx = \int \overline{\widehat{w_{pb}}(\xi)} \hat{f}(\xi)d\xi.$$

Definition: Local Wavevector Estimation

The local wavevector estimation of a function $f(x)$ at (p, b) is

$$v_f(p,b) = \frac{\nabla_b W_f(p,b)}{2\pi i W_f(p,b)}$$

for $p, b \in R^2$ such that $W_f(p,b) \neq 0$.

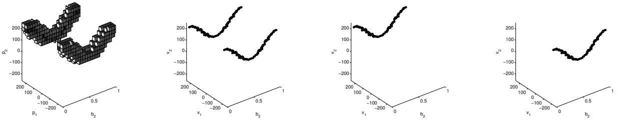
Definition: Synchrosqueezed Energy Distribution

Given $f(x)$, for $v, b \in R^2$, $W_f(p,b)$, and $v_f(p,b)$, the synchrosqueezed energy distribution $T_f(v,b)$ is

$$T_f(v,b) = \int |W_f(p,b)|^2 \delta(v_f(p,b) - v)dp.$$

Remark: The support of $T_f(v,b)$ is concentrating around local wavevectors.

Multiple components: $f(x) = \alpha_1(x)e^{2\pi i N\phi_1(x)} + \alpha_2(x)e^{2\pi i N\phi_2(x)}$.



1. Compute $W_f(p,b)$ and $\nabla_b W_f(p,b)$;
2. Compute $v_f(p,b)$ and perform synchrosqueezing to get $T_f(v,b)$;
3. Apply clustering and identify the disjoint supports of $T_f(v,b)$;
4. Reconstruct each intrinsic mode function using the dual frame.

$$f_i(x) = \int_{v_f(p,b)\in U_i} \tilde{w}_{pb}(x) W_f(p,b)\,dpdb,$$

where $\{\tilde{w}_{pb}(x), p, b \in R^2\}$ is a dual frame of $\{w_{pb}(x), p, b \in R^2\}$

.

Fast algorithms for Step 1 and 4. $O(L^2 \log L)$ for a $L \times L$ image.

Question: How well does this method work?

Theorem [Yang-Ying 12]

For a function $f(x) = \sum_{k=1}^{K} \alpha_k(x) e^{2\pi i N\phi_k(x)}$ and $\epsilon > 0$, we define

$$R_{f,\epsilon} = \{(p,b) : |W_f(p,b)| \geq |p|^{-s}\sqrt{\epsilon}\}$$

and

$$Z_{f,k} = \{(p,b) : |p - N\nabla\phi_k(b)| \leq |p|^s\}$$

for $1 \leq k \leq K$. There exists a constant $\epsilon_0(K) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there exists a constant $N_0(K,\epsilon) > 0$ such that for any $N > N_0(K,\epsilon)$ the following statements hold.

(i) $\{Z_{f,k} : 1 \leq k \leq K\}$ are disjoint and $R_{f,\epsilon} \subset \bigcup_{1\leq k\leq K} Z_{f,k}$;

(ii) For any $(p,b) \in R_{f,\epsilon} \cap Z_{f,k}$,

$$\frac{|v_f(p,b) - N\nabla\phi_k(b)|}{|N\nabla\phi_k(b)|} \lesssim \sqrt{\epsilon}.$$

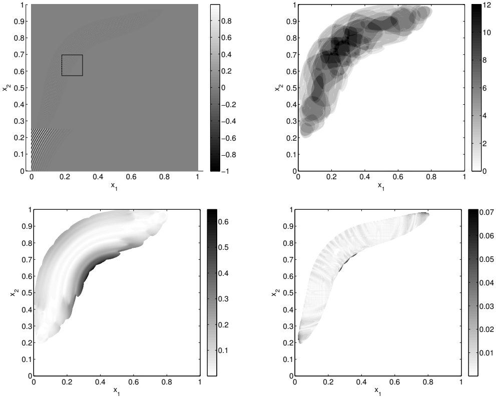


Figure : Top-left: A banded deformed plane wave. Top-right: Number of nonzero discrete synchrosqueezed energy of SSWPT. Bottom-left: Relative error of local wave-vector estimates using SSWPT. Bottom-right: Relative error of local wave-vector estimates using SSCT.

General curvelet transform: some notations

1. The scaling matrix
$$A_a = \begin{pmatrix} a^t & 0 \\ 0 & a^s \end{pmatrix},$$
where a is the distance from the center of one curvelet to the origin of Fourier domain.
2. The rotation angle θ and rotation matrix
$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
3. The unit vector $e_\theta = (\cos\theta, \sin\theta)^T$ of rotation angle θ.
4. θ_α represents the argument of given vector α.

Definition: 2D general curvelets

For $\frac{1}{2} < s < t < 1$, define

$$\widehat{w_{a\theta b}}(\xi) = \widehat{w}(A_a^{-1} R_\theta^{-1}(\xi - a \cdot e_\theta))e^{-2\pi i b\cdot\xi} a^{-\frac{t+s}{2}}$$

as a general curvelet in the Fourier domain. Equivalently,

$$w_{a\theta b}(x) = a^{\frac{t+s}{2}} e^{2\pi i a(x-b)\cdot e_\theta} w(A_a R_\theta^{-1}(x-b)).$$

In such a way, a family of curvelets
$\{w_{a\theta b}(x), a \in [1,\infty), \theta \in [0, 2\pi), b \in R^2\}$ is constructed.

Definition: 2D general curvelet Transform

$$W_f(a,\theta,b) = \int_{R^2} \overline{w_{a\theta b}(x)} f(x) dx$$

for $a \in [1,\infty)$, $\theta \in [0,2\pi)$, $b \in R^2$.

Definition: Local Wavevector Estimation

Local wave-vector estimation at (a,θ,b) is

$$v_f(a,\theta,b) = \frac{\nabla_b W_f(a,\theta,b)}{2\pi i W_f(a,\theta,b)} \tag{6}$$

for $a \in [1,\infty)$, $\theta \in [0,2\pi)$, $b \in R^2$ such that $W_f(a,\theta,b) \neq 0$.

Definition: Synchrosqueezed Energy Distribution

Synchrosqueezed energy distribution $T_f(v,b)$ is

$$T_f(v,b) = \int |W_f(a,\theta,b)|^2 \delta(\Re v_f(a,\theta,b) - v) adad\theta \tag{7}$$

for $v \in R^2$, $b \in R^2$.

Question: How well does this method work?

Theorem [Yang-Ying 13]

For a function $f(x) = \sum_{k=1}^{K} e^{-(\phi_k(x)-c_k)^2/\sigma_k^2}\alpha_k(x)e^{2\pi i N\phi_k(x)}$ and any $\epsilon > 0$, define

$$R_{f,\epsilon} = \left\{(a,\theta,b) : |W_f(a,\theta,b)| \geq a^{-\frac{s+t}{2}}\sqrt{\epsilon}\right\} \tag{8}$$

and

$$Z_{f,k} = \left\{(a,\theta,b) : |A_a^{-1}R_\theta^{-1}(a\cdot e_\theta - N\nabla\phi_k(b))| \leq 1\right\}$$

for $1 \leq k \leq K$. For fixed K, and any ϵ, there exists $N_0(K,\epsilon) > 0$ such that for any $N > N_0(K,\epsilon)$ the following statements hold.

(i) $\{Z_{f,k} : 1 \leq k \leq K\}$ are disjoint and $R_{f,\epsilon} \subset \bigcup_{1\leq k\leq K} Z_{f,k}$;

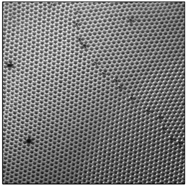
(ii) For any $(a,\theta,b) \in R_{f,\epsilon} \cap Z_{f,k}$,

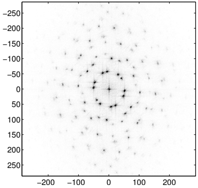
$$\frac{|v_f(a,\theta,b) - N\nabla\phi_k(b)|}{|N\nabla\phi_k(b)|} \lesssim \sqrt{\epsilon}.$$

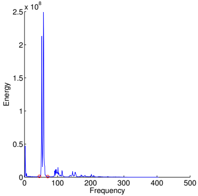
2D general intrinsic mode functions:

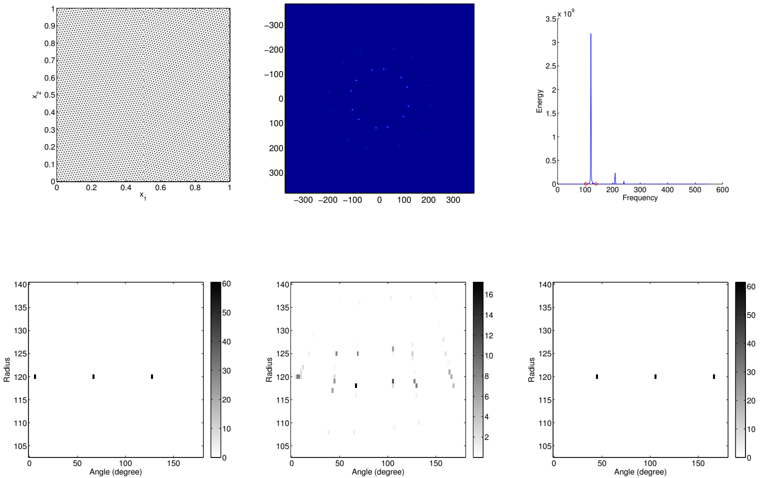
$$f(x) = \sum_{k=1}^{M} \chi_{\Omega_k}(x)\left(\alpha_k(x) S_k\left(2\pi N_k F_k \phi_k(x)\right)\right).$$

Mode segmentation problem:








x1
x2
Energy
Frequency
x 10^9
Radius
Angle (degree)

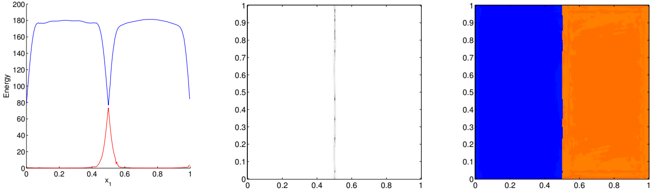


Figure : Left: The total enrgy function $TE_1(b)$ in blue and $TE_2(b)$ in red for fixed $b = 0.5$. Middle: The boundary indicator function $BD(b)$. Right: The weighted average angle $Angle(b)$ as an approximation of crystal rotations. The real crystal rotations are 15 degrees on the left and 57.5 degrees on the right.

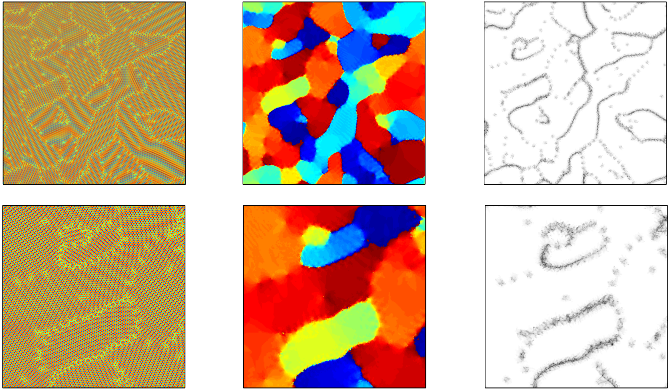


Figure : Left: A crystal image and its zoomed-in image. Middle: The crystal rotation and its zoomed-in result. Right: The crystal defects and its zoomed-in result.

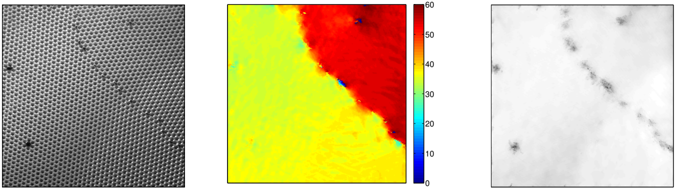


Figure : Left: A photograph of a bubble raft with strong reflection and point dislocations. Middle and right: rotations and boundaries.

Conclusion

- Developed several types of 1D and 2D synchrosqueezed transforms to analyze signals and images.
- Need more mathematical understanding and robust analysis.
- Fast algorithms for computing high dimensional transforms. Paralell computing? Distributed memory?