Synchrosqueezed Curvelet Transforms for 2D mode decomposition

Haizhao Yang

Department of Mathematics, Stanford University

Collaborators: Lexing Ying† and Jianfeng Lu♯

† Department of Mathematics and ICME, Stanford University

♯ Department of Mathematics, Duke University

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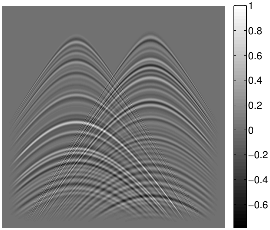
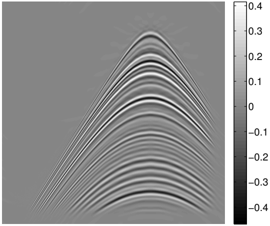
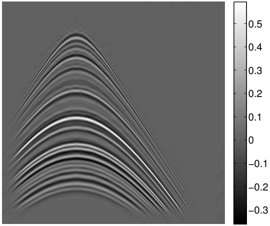
Geophysics

- A superposition of several wave-like components.
- Wave field separation and ground roll removal problems.







Materials science

Atomic crystal analysis

- Observation: an assemblage of wave-like components;
- Goal: Crystal segmentations, crystal rotations, crystal defects, crystal deformations.

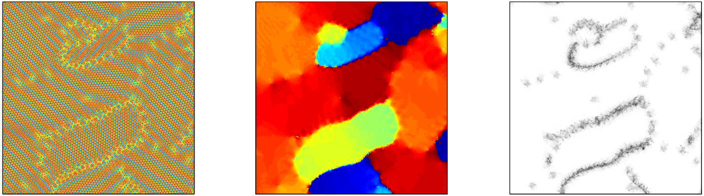


Figure : Left: An atomic crystal image. Middle: Estimated crystal rotation. Right: Identified crystal defects. Takes about 10s for a 1024*1024 image, while other methods using GPU computing take at least 40s.

1D mode decomposition

Known: A superposition of wave-like components

$$f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) e^{2\pi i N_k \phi_k(t)}.$$

Unknown: Number K, components $f_k(t)$, smooth instantaneous amplitudes $\alpha_k(t)$, smooth instantaneous frequencies $N_k \phi_k'(t)$.

Existing methods:

- Empirical mode decomposition methods (Huang et al. 98, 09);
- Synchrosqueezed wavelet transform (Daubechies et al. 09, 11); Synchrosqueezed wave packet transform (Y. 13);
- Data-driven time-frequency analysis (Hou et al. 11, 12, 13);
- Regularized nonstationary autoregression (Fomel 13);

1D synchrosqueezed wavelet transform (SSWT)

Continuous wavelet transform of a signal $s(t)$:

$$W_s(a,b) = \langle s(t), \phi_{ab}(t) \rangle = \int s(t) a^{-1/2} \overline{\phi(\frac{t-b}{a})}\, \mathrm{d}t.$$

EX: $s(t) = A\cos(\omega t)$.

$$W_s(a,b) \quad = \quad \frac{1}{2\pi} \int \widehat{s}(\xi) a^{1/2} \overline{\widehat{\phi}(a\xi)} e^{ib\xi}\, \mathrm{d}\xi = \frac{A}{4\pi} a^{1/2} \overline{\widehat{\phi}(a\omega)} e^{ib\omega}.$$

Synchrosqueezing for better readability

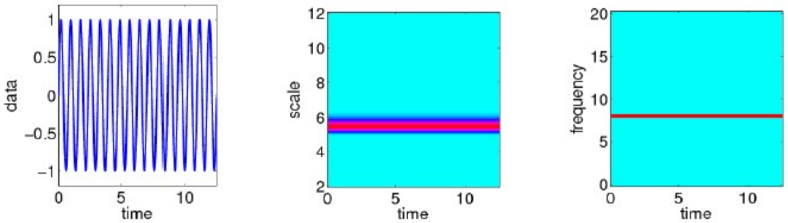


Figure : Numerical examples by Daubechies et al, signal $f(t) = sin(8t)$.

Definitions of SSWT

EX: $s(t) = A\cos(\omega t)$.

$W_s(a,b) = \frac{A}{4\pi} a^{1/2} \overline{\widehat{\phi}(a\omega)} e^{ib\omega} \Rightarrow \frac{\partial_b W_s(a,b)}{iW_s(a,b)} = \omega$

Definition: Instantaneous frequency estimate

$$\omega_s(a,b) = \frac{\partial_b W_s(a,b)}{iW_s(a,b)}.$$

Definition: Synchrosqueezed wavelet transform

$$\mathcal{T}_s(\omega,b) = \int_{\{a: W_s(a,b)\neq 0\}} W_s(a,b) a^{-3/2} \delta(\omega_s(a,b) - \omega)\,\mathrm{d}a.$$

Theory of SSWT

Theorem: (Daubechies, Lu, Wu 11 ACHA)

If

$$f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) e^{2\pi i N_k \phi_k(t)}$$

and $f_k(t)$ are well-separated, then

- $\mathcal{T}_f(a, b)$ has well-separated supports Z_k concentrating $(N_k \phi_k'(b), b)$;
- $f_k(t)$ can be accurately recovered by applying an inverse transform on $\mathcal{I}_{Z_k}(a, b)\mathcal{T}_f(a, b)$.

where $\mathcal{I}_{Z_k}(a, b)$ is an indication function.

2D mode decomposition

Known: A superposition of wave-like components

$$f(x) = \sum_{k=1}^{K} f_k(x) = \sum_{k=1}^{K} \alpha_k(x) e^{2\pi i N_k \phi_k(x)}.$$

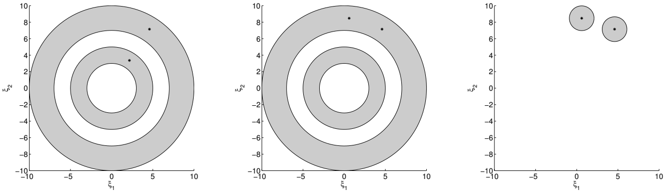
Unknown: Number K, components $f_k(x)$, local amplitudes $\alpha_k(x)$, local wave vectors $N_k \nabla \phi_k(x)$.

Existing methods:

1. 2D EMD methods (Huang et al.);
2. 2D Empirical wavelet, curvelet transforms (Gilles, Tran, Osher 13);
3. 2D SS wave packet and curvelet transforms (Y. and Ying 13, 14).

2D wavelet transform or wave packet transform?

Consider two plane waves $e^{ip\cdot x}$ and $e^{iq\cdot x}$ again.



- Left: Supports of continuous wavelets and plane waves with different wave numbers in the Fourier domain. Radial separation.
- Middle: Supports of wavelets and plane waves with the same wave number. No angular separation.
- Right: Supports of wave packets and plane waves with the same wave number. Angular separation.

Definition: Wave Packets
Given the mother wave packet $w(x)$ and $s \in (1/2, 1)$, the family of wave packets $\{w_{pb}(x), p, b \in R^2\}$ is defined as

$$w_{pb}(x) = |p|^s w(|p|^s(x - b))e^{2\pi i(x-b)\cdot p},$$

or equivalently in Fourier domain

$$\widehat{w_{pb}}(\xi) = |p|^{-s} e^{-2\pi i b\cdot\xi} \hat{w}(|p|^{-s}(\xi - p)).$$

Definition: The 2D wave packet transform of a function $f(x)$ is a function of $p, b \in R^2$

$$W_f(p, b) = \int \overline{w_{pb}(x)} f(x)\, \mathrm{d}x.$$

Definition: The local wavevector estimation of a function $f(x)$ at (p, b) is

$$v_f(p, b) = \frac{\nabla_b W_f(p, b)}{2\pi i W_f(p, b)}$$

for $p, b \in R^2$ such that $W_f(p, b) \neq 0$.

Definition: Given $f(x)$, for $v, b \in R^2$, $W_f(p, b)$, and $v_f(p, b)$, the synchrosqueezed energy distribution $T_f(v, b)$ is

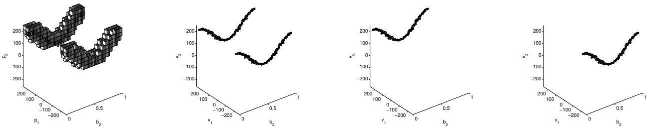
$$T_f(v, b) = \int_{\{p : W_f(p,b) \neq 0\}} |W_f(p, b)|^2 \delta(v_f(p, b) - v)\, \mathrm{d}p.$$

Remark: The support of $T_f(v, b)$ is concentrating around local wavevectors.

Sketch of 2D mode decomposition

Two components:

$$f(x) = \alpha_1(x)e^{2\pi iN\phi_1(x)} + \alpha_2(x)e^{2\pi iN\phi_2(x)}.$$



Fast algorithms for forward and inverse transforms. $O(L^2 \log L)$ for a $L \times L$ image.

Theory of 2D SS wave packet transform (SSWPT)

Theorem [Y.,Ying 13, SIAM Imaging Science]

For a well-separated superposition $f(x) = \sum_{k=1}^{K} \alpha_k(x) e^{2\pi i N \phi_k(x)}$ and $\epsilon > 0$, we define

$$R_{f,\epsilon} = \{(p, b) : |W_f(p, b)| \geq |p|^{-s}\sqrt{\epsilon}\}$$

and

$$Z_{f,k} = \{(p, b) : |p - N\nabla\phi_k(b)| \leq |p|^s\}$$

for $1 \leq k \leq K$. There exists a constant $\epsilon_0(K) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there exists a constant $N_0(K, \epsilon) > 0$ such that for any $N > N_0(K, \epsilon)$ the following statements hold.

(i) $\{Z_{f,k} : 1 \leq k \leq K\}$ are disjoint and $R_{f,\epsilon} \subset \bigcup_{1\leq k\leq K} Z_{f,k}$;

(ii) For any $(p, b) \in R_{f,\epsilon} \cap Z_{f,k}$,

$$\frac{|v_f(p, b) - N\nabla\phi_k(b)|}{|N\nabla\phi_k(b)|} \lesssim \sqrt{\epsilon}.$$

Banded wave-like components

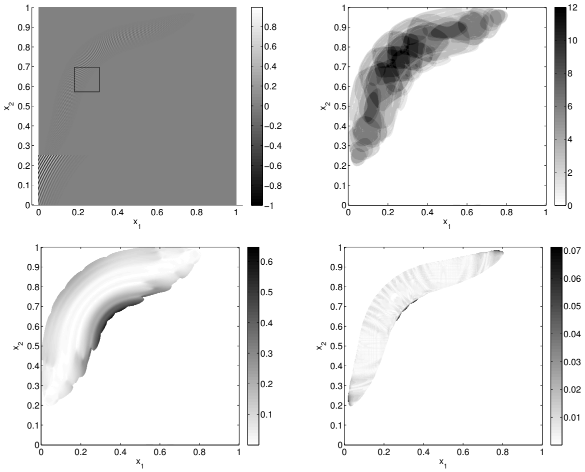


Figure : Top-left: A banded deformed plane wave. Top-right: Number of wave packet coefficients $|W_f(a,b)| > \delta$. Bottom-left: Relative error of local wave-vector estimates using SSWPT. Bottom-right: Relative error of local wave-vector estimates using SSCT.

General curvelet transform: Some notations

1. The scaling matrix
$$A_a = \begin{pmatrix} a^t & 0 \\ 0 & a^s \end{pmatrix}.$$
2. The rotation angle θ and rotation matrix
$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
3. The unit vector $e_\theta = (\cos\theta, \sin\theta)^T$ of rotation angle θ.
4. θ_α represents the argument of given vector α.

Definition: 2D general curvelets

For $\frac{1}{2} < s < t < 1$, define
$$w_{a\theta b}(x) = a^{\frac{t+s}{2}} e^{2\pi i a (x-b)\cdot e_\theta} w(A_a R_\theta^{-1}(x-b)).$$
A family of curvelets $\{w_{a\theta b}(x), a \in [1,\infty), \theta \in [0, 2\pi), b \in R^2\}$.

Definition: 2D general curvelet transform

$$W_f(a,\theta,b) = \int_{R^2} \overline{w_{a\theta b}(x)} f(x)\,\mathrm{d}x$$

for $a \in [1,\infty)$, $\theta \in [0,2\pi)$, $b \in R^2$.

Definition: local wave vector estimation

Local wave-vector estimation at (a,θ,b) is

$$v_f(a,\theta,b) = \frac{\nabla_b W_f(a,\theta,b)}{2\pi i W_f(a,\theta,b)}$$

for $a \in [1,\infty)$, $\theta \in [0,2\pi)$, $b \in R^2$ such that $W_f(a,\theta,b) \neq 0$.

Definition: synchrosqueezed energy distribution

Synchrosqueezed energy distribution $T_f(v,b)$ is

$$T_f(v,b) = \int_{\{(a,\theta):W_f(a,\theta,b)\neq 0\}} |W_f(a,\theta,b)|^2 \delta(\Re v_f(a,\theta,b) - v) a\,\mathrm{d}a\,\mathrm{d}\theta$$

for $v \in R^2$, $b \in R^2$.

Theory of the SS curvelet transform (SSCT)

Theorem [Y., Ying SIAM Mathematical Analysis 14] Suppose $\frac{1}{2} < s < \eta < t$ are fixed, and a well-separated superposition

$$f(x) = \sum_{k=1}^{K} e^{-(\phi_k(x)-c_k)^2/\sigma_k^2} \alpha_k(x) e^{2\pi i N \phi_k(x)},$$

where $\sigma_k \geq N^{-\eta}$. For any $\epsilon > 0$, define

$$R_{f,\epsilon} = \left\{(a,\theta,b) : |W_f(a,\theta,b)| \geq a^{-\frac{s+t}{2}}\sqrt{\epsilon}\right\}$$

and

$$Z_{f,k} = \left\{(a,\theta,b) : |A_a^{-1} R_\theta^{-1}(a \cdot e_\theta - N\nabla\phi_k(b))| \leq 1\right\}$$

for $1 \leq k \leq K$. For any ϵ, there exists $N_0(\epsilon) > 0$ such that for any $N > N_0(\epsilon)$ the following statements hold.

(i) $\{Z_{f,k} : 1 \leq k \leq K\}$ are disjoint and $R_{f,\epsilon} \subset \bigcup_{1\leq k\leq K} Z_{f,k}$;

(ii) For any $(a,\theta,b) \in R_{f,\epsilon} \cap Z_{f,k}$,

$$\frac{|v_f(a,\theta,b) - N\nabla\phi_k(b)|}{|N\nabla\phi_k(b)|} \lesssim \sqrt{\epsilon}.$$

Applications in Geophysics

Noisy Example 1 for SS curvelet transform

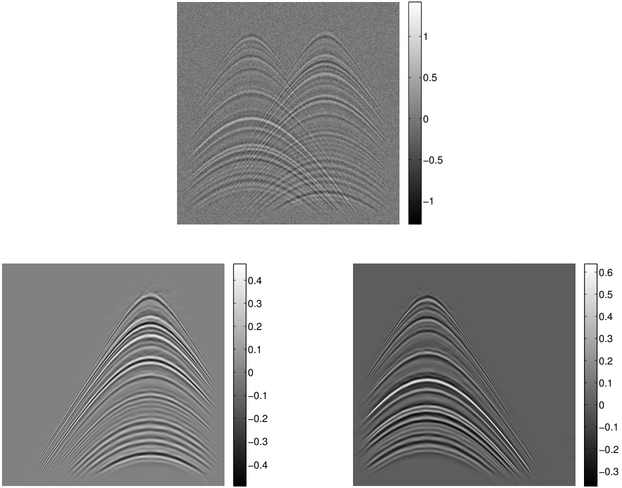


Figure : Top: A noisy superposition of two components. Left: First recovered component. Right: Second recovered component.

Noisy Example 2 for SS curvelet transform

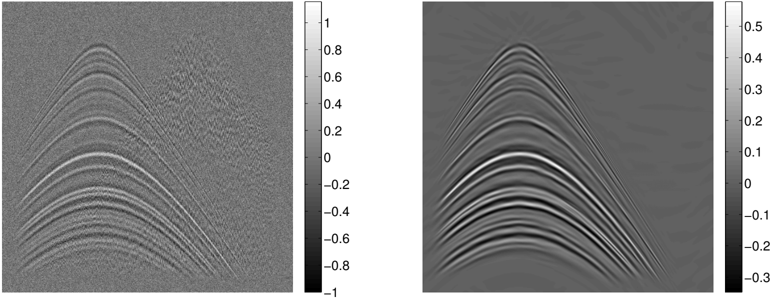


Figure : Left: A noisy superposition of two components. Right: The recovered main component.

A real example

comment

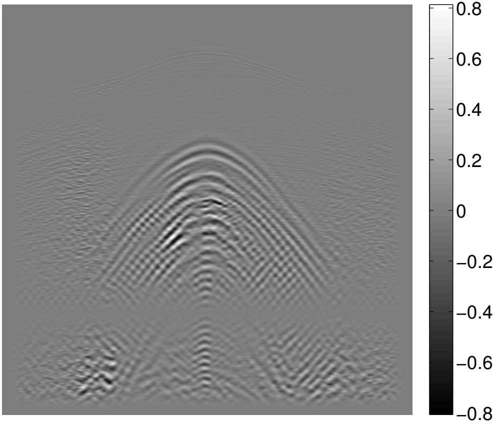


Figure : A superposition of several components.

Results of the real example

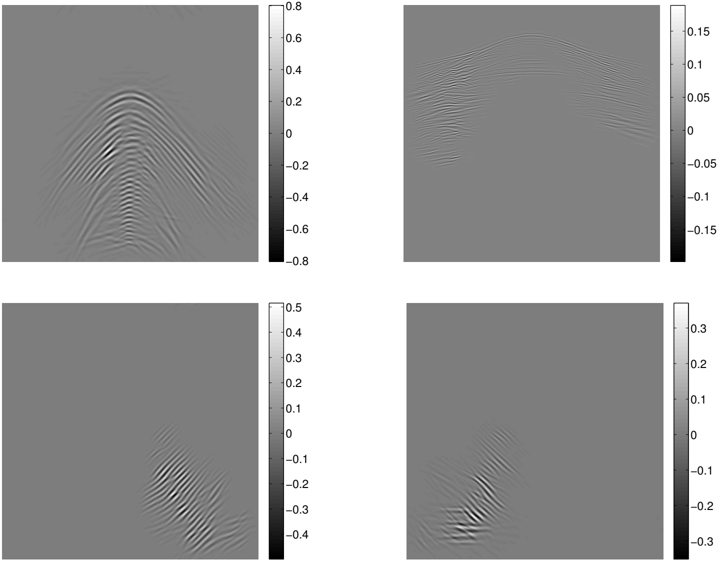


Figure : Four recovered components.

Toy example of an atomic crystal image

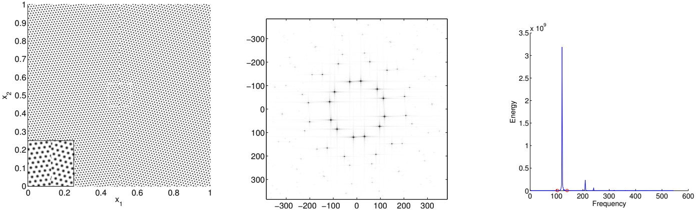


Figure : Left: Crystal image. Crystal rotations, 7.5 degrees on the left and 45 on the right. Middle: 2D Fourier power spectrum. Right: Radially average Fourier power spectrum.

Results of this toy example

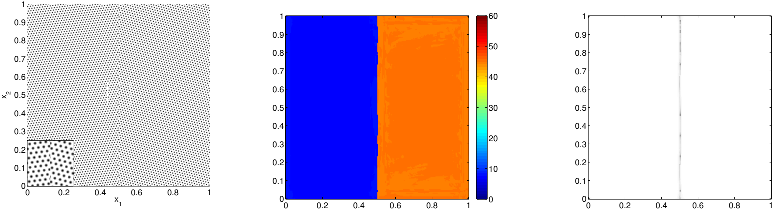


Figure : Left: Crystal image. Middle: The angle estimates of crystal rotations. Right: The boundary indicator function.

Phase field crystal image

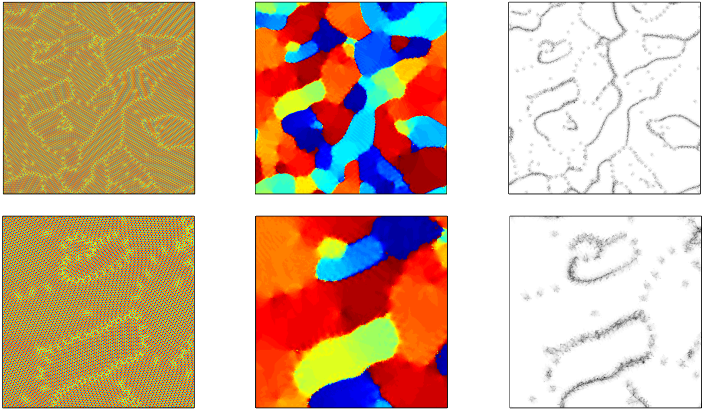


Figure : Left: A crystal image and its zoomed-in image. Middle: The crystal rotation and its zoomed-in result. Right: The crystal defects and its zoomed-in result. Takes about 10s, while other methods take at least 40s.

Conclusion

Developed 2D synchrosqueezed transforms to analyze images.

- A method with rigorous mathematical analysis.
- A method with both radial and angular separations.
- The SS curvelet transform can adapt different data structure: s and t.
- Fast algorithms to compute the forward and inverse transforms.
- Succesful applications in geophysics and materials science.

Results of 2D EEMD

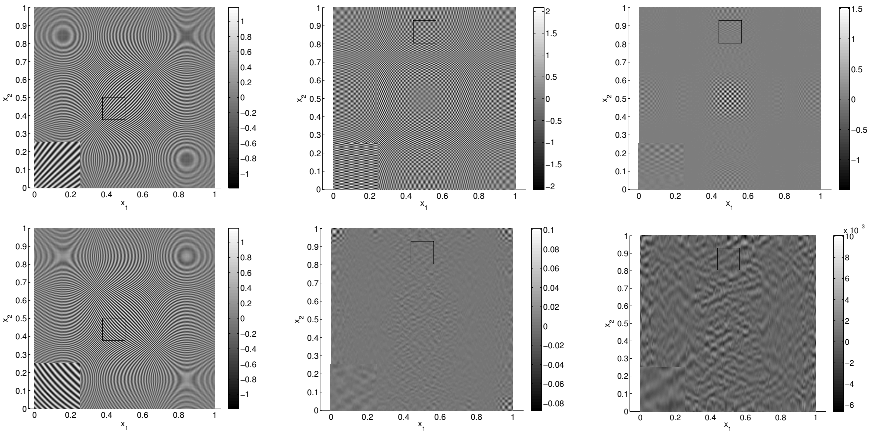
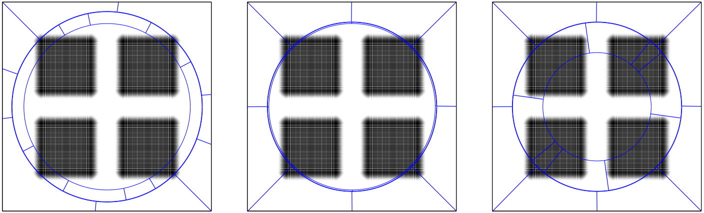


Figure : A superposition of two components with similar local wave number

Results of 2D Empirical curvelet transforms



2D Fourier power spetrum of a superposition of two components and Fourier domain partitions using empirical curvelet transforms.