Synchrosqueezed Wave Packet Transforms and Diffeomorphism Based Spectral Analysis for 1D General Mode Decompositions

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Data Analysis:

- Superposition of several patterns
- Differ in instantaneous frequency
- Nonlinear and non-stationary with sudden changes

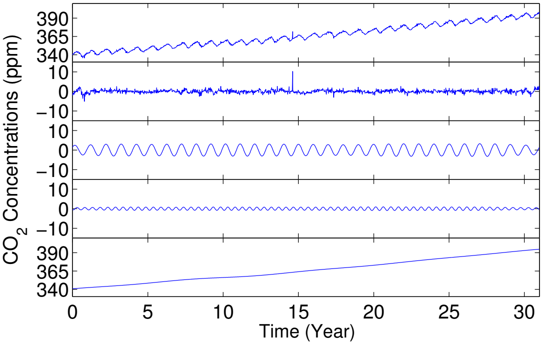


Figure : Decomposition using wavelets

Mode decomposition

Known: A superposition of wave-like components

$$f(x) = \sum_{k=1}^{K} f_k(x) = \sum_{k=1}^{K} \alpha_k(x) e^{2\pi i N_k \phi_k(x)}.$$

Unknown:

- Number K
- Components $f_k(x)$
- Smooth instantaneous amplitudes $\alpha_k(x)$
- Smooth instantaneous frequencies $N_k \phi'_k(x)$

General mode decomposition problem

Known: A superposition of general wave-like components

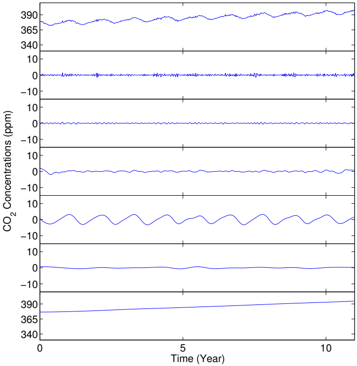
$$f(x) = \sum_{k=1}^{K} f_k(x) = \sum_{k=1}^{K} \alpha_k(x) s_k(2\pi N_k \phi_k(x)),$$

Unknown:

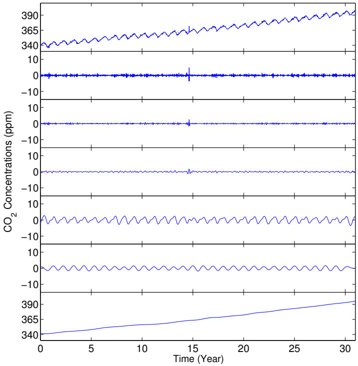
- Basic unknown information
- General wave shape functions $s_k(x)$, i.e. $\{(n, \widehat{s}_k(n) : \widehat{s}_k(n) \neq 0\}$

Empirical mode decomposition (Huang et al. 98,99,09)

- More physical meaning
- Better than Fourier and wavelet analysis
- Need more mathematical understanding



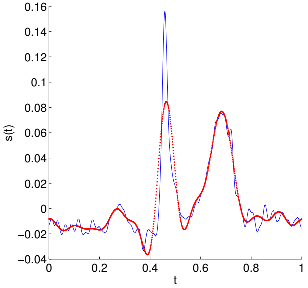
Useful decomposition



Misleading decomposition

Synchrosqueezed wavelet transform (Daubechies et al. 95 11)

- Band-limited wave shape function s_k (Wu 2012)
- No explicit statement about the reconstruction of s_k.



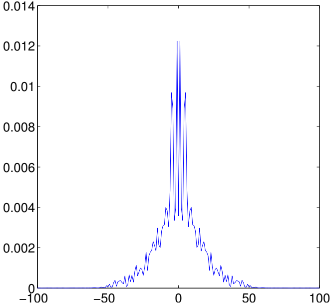
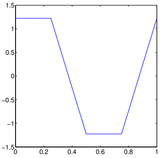


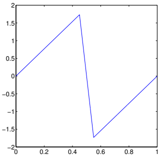
Figure : Left: Real Echocardiography (ECG) wave shape and its band-limited approximation. Right: Fourier power spectrum.

Difficulties in general mode decomposition

- Need accurate estimates of high frequency information
- Crossover frequencies, poor scale separation



s_1



s_2

$$
\begin{aligned}
& s_1(2\pi N_1\phi_1(t)) + s_2(2\pi N_2\phi_2(t)) \\
= & \sum_n \widehat{s}_1(n) e^{2\pi i n N_1\phi_1(t)} \\
& + \sum_n \widehat{s}_2(n) e^{2\pi i n N_2\phi_2(t)}
\end{aligned}
$$

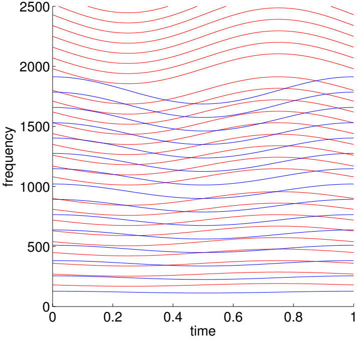


Figure : Instantaneous frequencies: s_1 in blue and s_2 in red.

Innovation 1

Synchrosqueezed wave packet transforms with a scaling parameter $s \in (1/2, 1)$

- Better resolution to estimate instantaneous information in high frequency domain
- General wave shapes s_k, NOT band-limited
- Fast algorithm $O(N \log N)$

Innovation 2

Diffeomorphism based spectral analysis (DSA)

- First method to reconstruct s_k
- Efficient implementation $O(KMN)$, M is the number of principal Fourier modes of all s_k and K is the number of s_k.

Synchrosqueezed wavelet transform (SSWT)

Continuous wavelet transform of a signal $s(t)$:

$$W_s(a,b) = \langle s(t), \phi_{ab}(t) \rangle = \int s(t) a^{-1/2} \overline{\phi(\frac{t-b}{a})} \, \mathrm{d}t.$$

EX: $s(t) = A\cos(\omega t)$.

$$W_s(a,b) \quad = \quad \frac{1}{2\pi} \int \widehat{s}(\xi) a^{1/2} \overline{\widehat{\phi}(a\xi)} e^{ib\xi} \, \mathrm{d}\xi = \frac{A}{4\pi} a^{1/2} \overline{\widehat{\phi}(a\omega)} e^{ib\omega}.$$

Synchrosqueezing for better readability

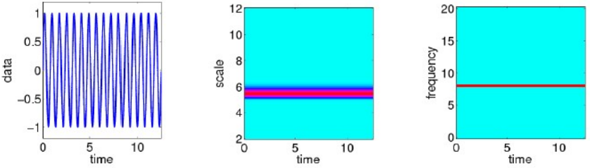


Figure : Numerical examples by Daubechies et al, signal $f(t) = sin(8t)$.

Definitions of SSWT

EX: $s(t) = A\cos(\omega t)$.

$W_s(a,b) = \frac{A}{4\pi} a^{1/2} \overline{\widehat{\phi}(a\omega)} e^{ib\omega} \Rightarrow \frac{\partial_b W_s(a,b)}{iW_s(a,b)} = \omega$

Definition: Instantaneous frequency estimate

$$\omega_s(a,b) = \frac{\partial_b W_s(a,b)}{iW_s(a,b)}.$$

Definition: Synchrosqueezed wavelet transform

$$\mathcal{T}_s(\omega,b) = \int_{\{a: W_s(a,b)\neq 0\}} W_s(a,b) a^{-3/2} \delta(\omega_s(a,b) - \omega)\, \mathrm{d}a.$$

Theory of SSWT

Theorem: (Daubechies, Lu, Wu 11 ACHA)

If

$$f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) e^{2\pi i N_k \phi_k(t)}$$

and $f_k(t)$ are well-separated, then

- $\mathcal{T}_f(a, b)$ has well-separated supports Z_k concentrating $(N_k \phi_k'(b), b)$;
- $f_k(t)$ can be accurately recovered by applying an inverse transform on $\mathcal{I}_{Z_k}(a, b)\mathcal{T}_f(a, b)$.

where $\mathcal{I}_{Z_k}(a, b)$ is an indication function.

Mode decomposition example by SSWT

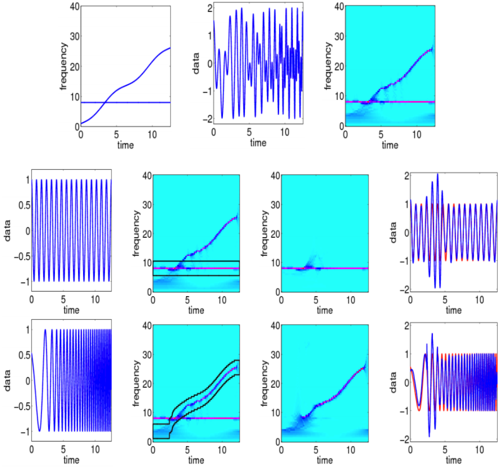


Figure : A superposition of two modes with **crossover** frequencies. Courtesy of [Daubechies, Lu, Wu 11].

Synchrosqueezed wave packet transform (SSWPT)

Wave packets $\{w_{ab}(t) : |a| \geq 1, b \in R\}$ with $s \in (1/2, 1)$:

$$w_{ab}(t) = |a|^{s/2} w(|a|^s (t-b)) e^{2\pi i (t-b) a},$$

or equivalently, in the Fourier domain as

$$\widehat{w_{ab}}(\xi) = |a|^{-s/2} e^{-2\pi i b \xi} \widehat{w}(|a|^{-s}(\xi - a)).$$

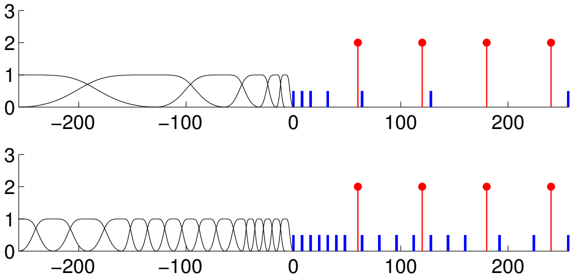


Figure : Comparison of tilings. Top: wavelets. Bottom: wave packets.

Definition: Instantaneous frequency estimate

$$\omega_s(a,b) = \frac{\partial_b W_s(a,b)}{2\pi i W_s(a,b)}.$$

Definition: Synchrosqueezed wave packet transform

$$\mathcal{T}_s(\omega,b) = \int_{\{a: W_s(a,b)\neq 0\}} |W_s(a,b)|^2 \delta(\Re\omega_s(a,b)-\omega)\, \mathrm{d}a.$$

Theorem of SSWPT (Y. 13)

If

$$f(t)=\sum_{k=1}^{K} f_k(t)=\sum_{k=1}^{K}\alpha_k(t)s_k\left(2\pi N_k\phi_k(t)\right)$$

and $f_k(t)$ has a well-separated Fourier series mode $\widehat{s}_k(n)\alpha_k(t)e^{2\pi i n N_k\phi_k(t)}$, then

- $\mathcal{T}_f(a,b)$ has a well-separated support Z_{kn} concentrating $(nN_k\phi_k'(b),b)$;
- $\widehat{s}_k(n)\alpha_k(t)e^{2\pi i n N_k\phi_k(t)}$ can be accurately recovered by applying an inverse transform on $\mathcal{I}_{Z_{kn}}(a,b)\mathcal{T}_f(a,b)$, where $\mathcal{I}_{Z_k}(a,b)$ is an indicator function.

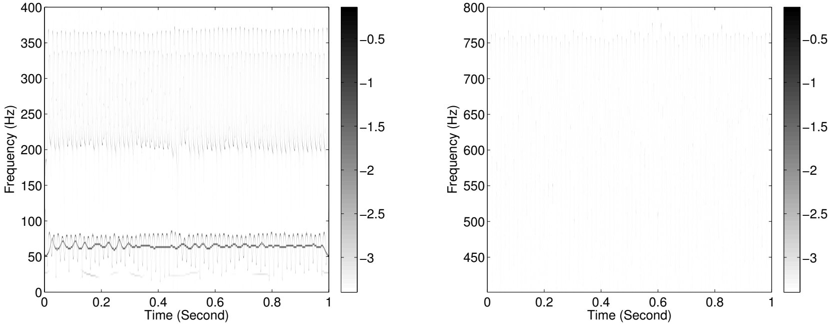


Figure : The SSWT of the same ECG signal. Left: low frequency part. Right: high frequency part.

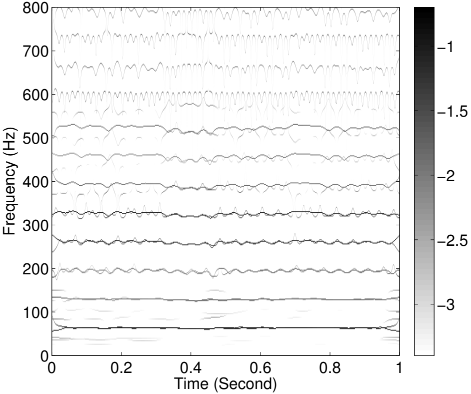
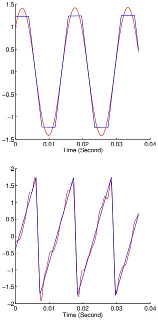
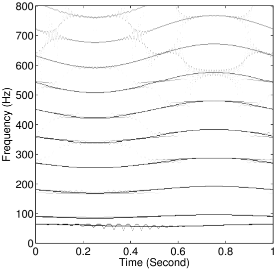


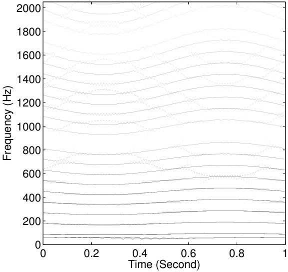
Figure : The SSWPT of an ECG signal.

General mode decomposition:

$$
\begin{aligned}
f(t) &= s_1(2\pi N_1\phi_1(t)) + s_2(2\pi N_2\phi_2(t)) \\
&= \sum_n \widehat{s}_1(n) e^{2\pi i n N_1 \phi_1(t)} + \sum_n \widehat{s}_2(n) e^{2\pi i n N_2 \phi_2(t)}
\end{aligned}
$$







Diffeomorphism based spectral analysis (DSA)

Suppose

$$f(t) = s_1(2\pi N_1 \phi_1(t)) + s_2(2\pi N_2 \phi_2(t)),$$

we know $p_k(t) = N_k \phi_k(t)$, $k = 1, 2$.
Goal: find n and $\widehat{s}_k(n)$ such that $\widehat{s}_k(n) \neq 0$.
Let

$$h_k(t) = f \circ p_k^{-1}(t) = \sum_{n=-\infty}^{\infty} \widehat{s}_k(n) e^{2\pi int} + \sum_{j \neq k} \sum_{n=-\infty}^{\infty} \widehat{s}_j(n) e^{2\pi i n N_j \tilde{\phi}_j(t)}.$$

Idea:

1. The location of the peak of $|\widehat{h}_k(\xi)|$ tell us one candidate n. Hence, solve
$$(\tau, j) = \arg\max_{(\xi,k)} |\widehat{h_k}(\xi)|.$$
2. Solve
$$\widehat{s}_j(\tau) = \arg\min_{\beta \in C} \|f(t) - \beta e^{2\pi i \tau N_j \phi_j(t)}\|_{L^2}.$$
3. Update $f(t) = f(t) - \widehat{s}_j(\tau) e^{2\pi i \tau N_j \phi_j(t)}$ and repeat this process.

Theory of DSA

Suppose $f(t) = \sum_{k=1}^{K} \alpha_k(t) s_k(2\pi N_k \phi_k(t))$ and phase functions $\{\phi_k(t)\}_{1 \le k \le K}$ are well-different. Define

$$h_k(t) = \frac{f \circ \phi_k^{-1}(t)}{\alpha_k \circ \phi_k^{-1}(t)}.$$

For a Gabor transform $\mathcal{F}_T$ with a sufficiently large window, let

$$(a_0, k_0) = \underset{(a,k)}{\arg\max} |\mathcal{F}_T(h_k)(a, b)|,$$

then $a_0 \approx nN_{k_0}$ for some n such that $\widehat{s_{k_0}}(n) \neq 0$.

General mode decomposition

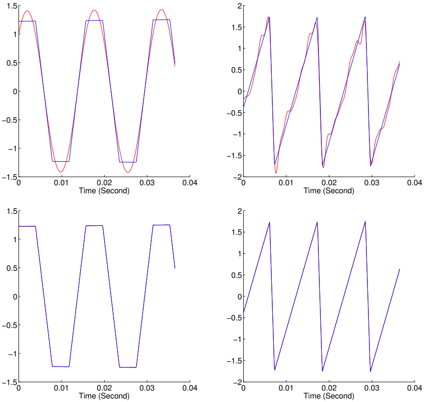
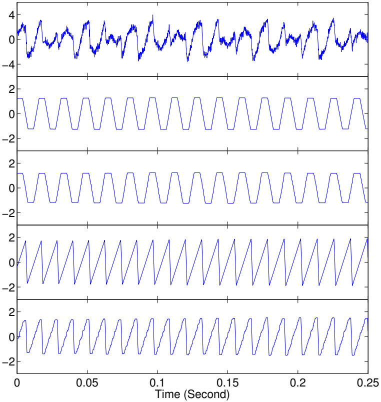


Figure : Blue: Real signals. Red: Reconstructed results. Top: Two recovered modes using a few well separated components. Bottom: Two recovered general modes provided by the DSA method.

Example 1



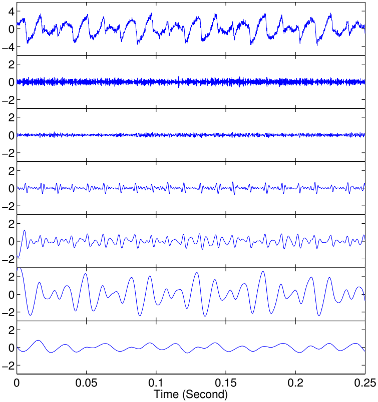


Figure : Toy Example. Left: SSWPT and DSA. Right: EEMD.

Example 2

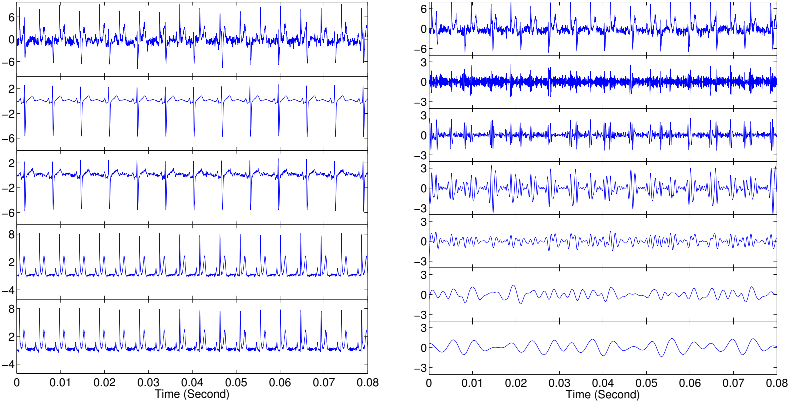


Figure : ECG spike shape function. Left:SSWPT and DSA. Right: EEMD.

Example 3

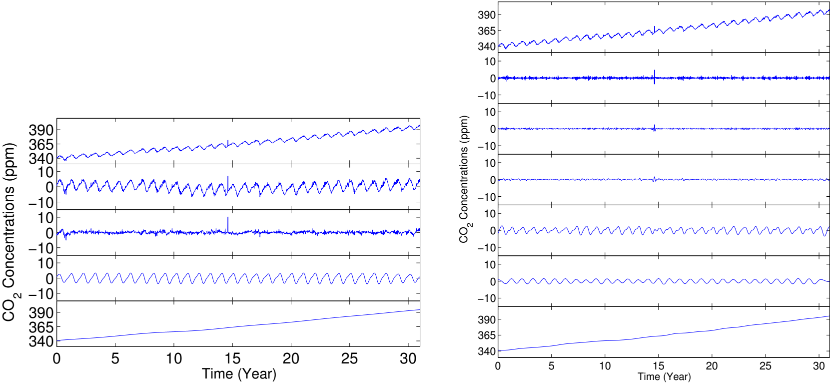


Figure : Real CO_2 concentration data. Left: SSWPT and DSA. Right: EEMD.

Thank you!