Synchrosqueezed wave packet transform for 2D mode decomposition

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1D Mode decomposition:

- Superposition of several paterns
- Nonlinear and non-stationary with jumps and sudden changes
- Analyze instantaneous oscillation

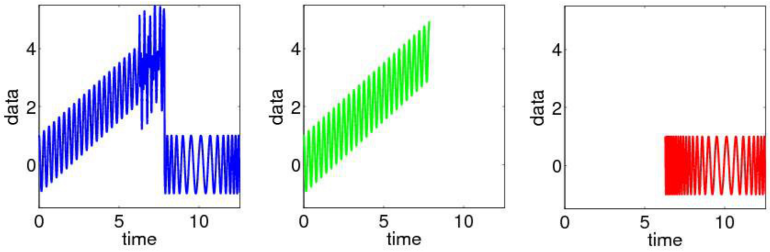


Figure : A superposition of two modes. Courtesy of [Daubechies-Lu-Wu 11].

Empirical mode decomposition and its variants

- Work of N. Huang, Z. Wu, T. Hou, ...
- Applied to a wide class of signals in science and engineering
- Lack of theoretical analysis

An optimization method

- Work of T. Hou and Z. Shi.
- Sparse representation in a data-driven time-frequency dictionary.
- Nonlinear L^0 optimization model and iterative nonlinear matching pursuit solver

The synchrosqueezed wavelet transform

- A case of reallocation methods [Auger, Chassande-Mottin, Daubechies, Flandrin,...]
- Developed in [Daubechies-Maes 96] and analyzed in [Daubechies-Lu-Wu 11].

2D mode decomposition

1. A superposition of several wave-like components with diferent oscillatory patterns.
2. Applications in acoustic and electromagnetic scattering, seismic imaging.
3. Differ in wavenumbers: e.g. $e^{i(x_1+x_2)}$ and $e^{i2(x_1+x_2)}$.
4. Differ in directions: e.g. $e^{i(x_1+x_2)}$ and $e^{i(x_1-x_2)}$.

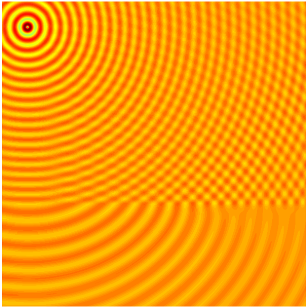


Figure : A solution of Helmoltz equation with a single point source in a layered medium.

2D EMD and its variants

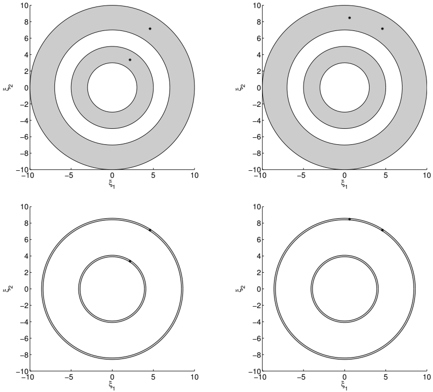
- Methods based on 2D interpolation and a similar sifting process in 1D EMD
- Methods based on mode decompositions of 1D data slices.

Quetion:

- Can they separate two modes $e^{ip\cdot x}$ and $e^{iq\cdot x}$?
  If $|p| \neq |q|$, Yes. If $|p| = |q|$, No.
- 2D interpolation based method:
  If $|p| = |q|$, $e^{ip\cdot x} + e^{iq\cdot x}$ is one intrinsic mode by definition.
- 1D decomposition based method:
  If $|p| = |q|$, each 1D data slice contains two modes with the same frequency.
  EMD fails because of the beating phenomenon. [Rilling, G. and Flandrin, P. 2008].

2D synchrosqueezed wavelet transform:

Consider two plane waves $e^{ip \cdot x}$ and $e^{iq \cdot x}$ again.



Top: Supports of continuous wavelets and plane waves in the Fourier domain.
Bottom: Supports of the synchrosqueezed transform.

Summary of existing methods

Consider two plane waves $e^{ip\cdot x}$ and $e^{iq\cdot x}$.

1. If $|p| \neq |q|$, all these methods can separate $e^{ip\cdot x} + e^{iq\cdot x}$, i.e., they have the radial separation.
2. If $|p| = |q|$ and $p \neq q$, all these methods fails, i.e., they lack the angular separation.

Question: How to obtain the angular separation?
Answer: Anisotropic synchrosqueezed wave packet transforms.

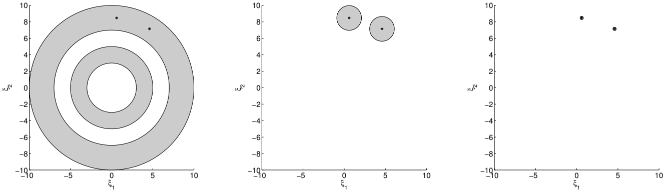


Figure : Left:suports of continuous wavelets. Middle: suports of wave packets. Right: synchrosqueezed wave packet transform.

Definition: Wave Packets

Given the mother wave packet $w(x)$ and $s \in (1/2, 1)$, the family of wave packets $\{w_{pb}(x), p, b \in R^2\}$ is defined as

$$w_{pb}(x) = |p|^s w(|p|^s(x - b))e^{2\pi i(x-b)\cdot p},$$

or equivalently in Fourier domain

$$\widehat{w_{pb}}(\xi) = |p|^{-s} e^{-2\pi i b\cdot\xi} \hat{w}(|p|^{-s}(\xi - p)).$$

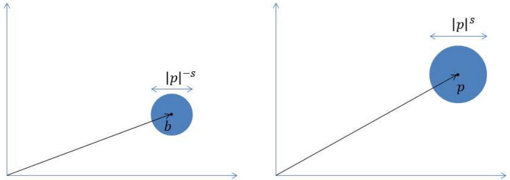


Figure : Left:the support of $w_{pb}(x)$ in spacial domain. Right: the support of $\widehat{w_{pb}}(\xi)$ in the Fourier domain.

Definition: Wave Packet Transform

The wave packet transform of a function $f(x)$ is a function of $p, b \in R^2$

$$W_f(p,b) = \int \overline{w_{pb}(x)} f(x) dx = \int \overline{\widehat{w_{pb}}(\xi)} \hat{f}(\xi) d\xi.$$

Definition: Local Wavevector Estimation

The local wavevector estimation of a function $f(x)$ at (p, b) is

$$v_f(p,b) = \frac{\nabla_b W_f(p,b)}{2\pi i W_f(p,b)}$$

for $p, b \in R^2$ such that $W_f(p,b) \neq 0$.

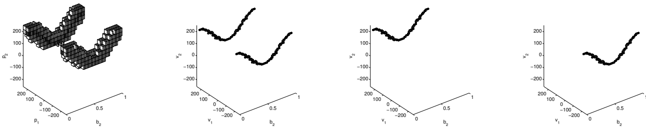
Definition: Synchrosqueezed Energy Distribution

Given $f(x)$, for $v, b \in R^2$, $W_f(p,b)$, and $v_f(p,b)$, the synchrosqueezed energy distribution $T_f(v,b)$ is

$$T_f(v,b) = \int |W_f(p,b)|^2 \delta(v_f(p,b) - v) dp.$$

Remark: The support of $T_f(v,b)$ is concentrating around local wavevectors.

Multiple components: $f(x) = \alpha_1(x)e^{2\pi iN\phi_1(x)} + \alpha_2(x)e^{2\pi iN\phi_2(x)}$.



1. Compute $W_f(p, b)$ and $\nabla_b W_f(p, b)$;
2. Compute $v_f(p, b)$ and perform synchrosqueezing to get $T_f(v, b)$;
3. Apply clustering and identify the disjoint supports of $T_f(v, b)$;
4. Reconstruct each intrinsic mode function using the dual frame.

$$f_i(x) = \int_{v_f(p,b)\in U_i} \tilde{w}_{pb}(x) W_f(p, b) dpdb,$$

where $\{\tilde{w}_{pb}(x), p, b \in R^2\}$ is a dual frame of $\{w_{pb}(x), p, b \in R^2\}$.

Fast algorithms for Step 1 and 4.$O(L^2 \log L)$ for a $L \times L$ image.

Question:How well does this method work?
Theorem [Yang-Ying 11]
For a function $f(x)$ and $\epsilon > 0$, we define

$$R_{f,\epsilon} = \{(p,b) : |W_f(p,b)| \geq |p|^{-s}\sqrt{\epsilon}\}$$

and

$$Z_{f,k} = \{(p,b) : |p - N\nabla\phi_k(b)| \leq |p|^s\}$$

for $1 \leq k \leq K$. For fixed M and K, there exists a constant $\epsilon_0(M,K) > 0$ such that for any $\epsilon \in (0,\epsilon_0)$ there exists a constant $N_0(M,K,\epsilon) > 0$ such that for any $N > N_0(M,K,\epsilon)$ and $f(x) \in F(M,N,K)$ the following statements hold.

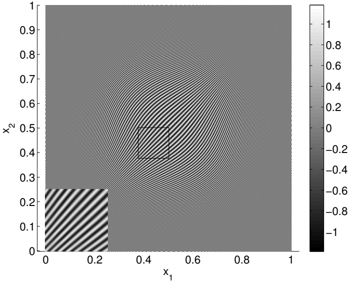
(i) $\{Z_{f,k} : 1 \leq k \leq K\}$ are disjoint and $R_{f,\epsilon} \subset \bigcup_{1\leq k\leq K} Z_{f,k}$;

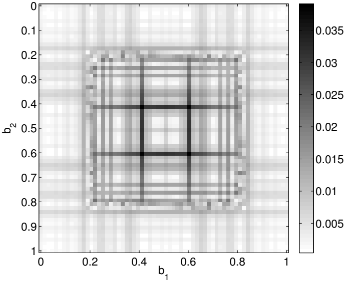
(ii) For any $(p,b) \in R_{f,\epsilon} \cap Z_{f,k}$,

$$\frac{|v_f(p,b) - N\nabla\phi_k(b)|}{|N\nabla\phi_k(b)|} \lesssim \sqrt{\epsilon}.$$

Local wavevecter extraction

Let $f(x) = \alpha(x)e^{2\pi i N\phi(x)}$ with $\alpha(x) = 1$,
$\phi(x) = \phi(x_1, x_2) = x_1 + x_2 + 0.1\sin(2\pi x_1) + 0.1\sin(2\pi x_2)$, and
$N = 135$. The relative approximation error is of order $O(\sqrt{\epsilon}) = 10^{-2}$



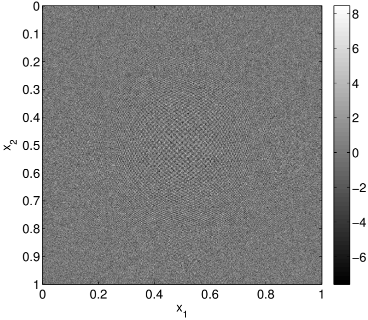


Noicy Signal of Two Deformed Plane Waves

Consider

$$f(x) = e^{2\pi i N \phi_1(x)} + e^{2\pi i N \phi_2(x)} + n(x),$$

where $n(x)$ is an isotropic complex Gaussian random noise with zero mean.



SNR	∞	3	0	-3
PSNR_1	54.19	24.86	22.87	18.78
PSNR_2	54.13	24.79	23.02	18.67

Table 1: SNR and PSNR for each mode

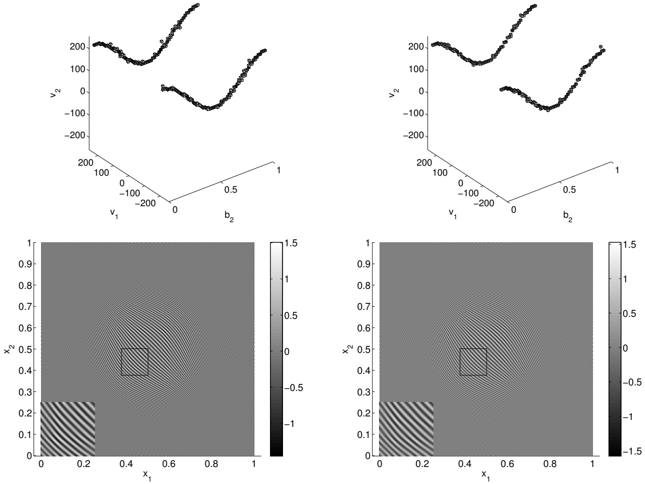


Figure : Left: $SNR = 0$. Right: $SNR = -3$. Second column: the second reconstructed mode. Run time: 265 seconds

MEEMD as a comparison:

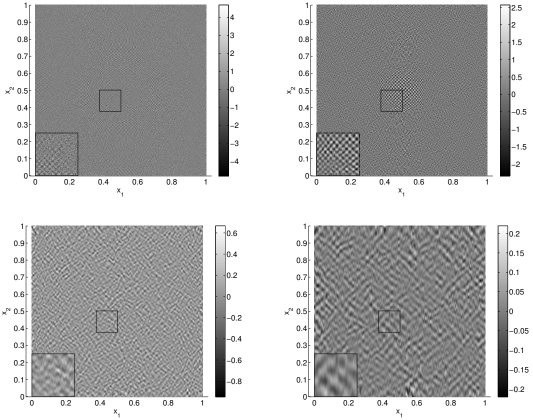
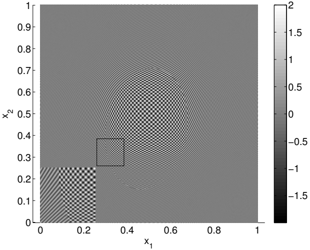
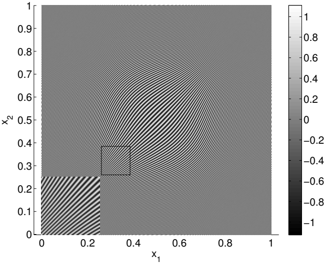
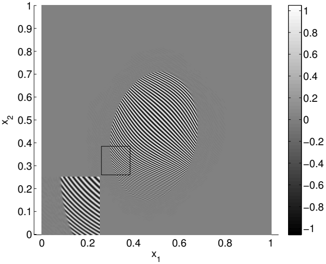


Figure : The ensemble number is 100, the noise ratio is 0.2, and the number of recovered modes in the code is 4. The run time of MEEMD is 16 hours and is dependent on the number of recovered modes set up a prior.

Signal of Incompleted Deformed Plane Waves







Real application:

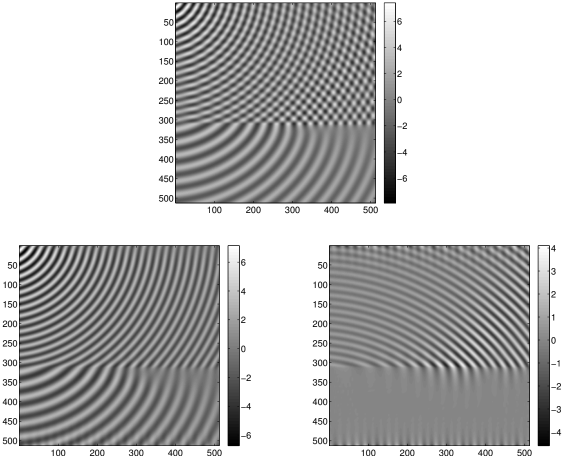


Figure : Seismic wavefield propagating through a layered media equation. Bottom: downward propagating wave and the upward reflected wave.

Conclusion:

- First methd with both the radial separation and angular separation.
- A wide range of well-separated nonlinear wave components can be extracted accurately with analytic and numerical proof.
- Robust against noise.

Future Work

- $s \in (1/2, 1)$ is required in current work. Question: Will it work for the wave atom case where $s = 1/2$?
- A synchrosqueezed curvelet transform would have better applications in some problems.
- Crossover local wavevectors. Most important problem to be solved.