Robustness Analysis of Synchrosqueezed Transforms

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Medical study (Yang, ACHA, 14)

- A superposition of two ECG signals.

$$f(t) = \alpha_1(t)s_1(2\pi\phi_1(t)) + \alpha_2(t)s_2(2\pi\phi_2(t)).$$

- Spike wave shape functions $s_1(t)$ and $s_2(t)$.

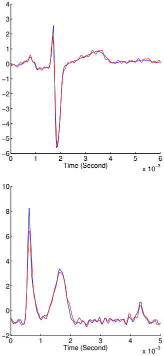


Figure : Complicated wave shape functions.

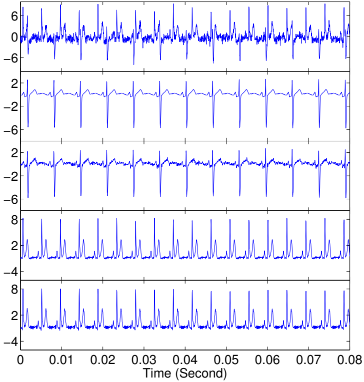


Figure : Good decomposition.

Geophysics (Yang and Ying, SIIMS 13, SIMA 14)

- A superposition of several wave fields.
- Nonlinear components, bounded supports.

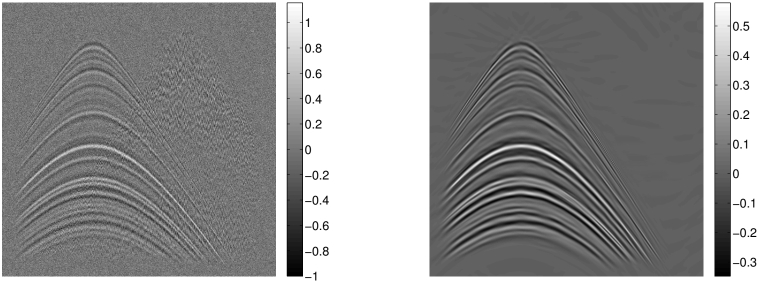
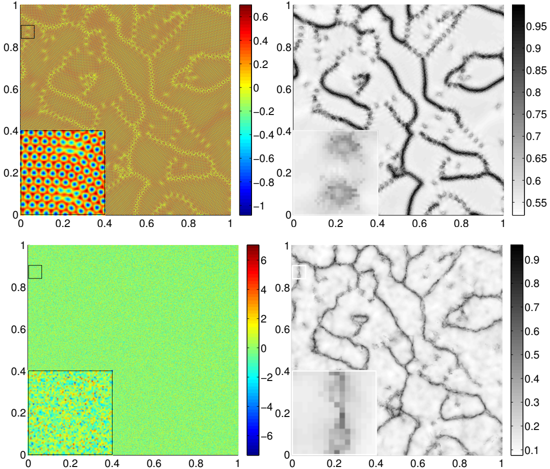


Figure : One target component with structure noise and Gaussian random noise. Courtesy of Fomel and Hu for providing data expames.

Materials science (Yang, Lu and Ying, preprint)

Atomic crystal analysis

- Observation: an assemblage of wave-like components;
- Goal: Crystal segmentation, crystal rotations, crystal defects, crystal deformations.



Art forensics (Yang, Lu, Daubechies, Ying, preprint)

Painting canvas analysis

- Observation: a superposition of wave-like components;
- Goal: count threads and estimate texture deformation.

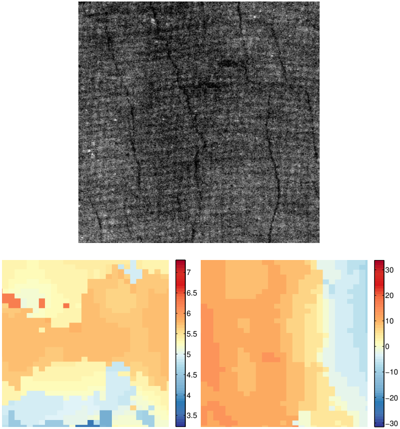


Figure : Top: a X-ray image of canvas. Left: horizontal thread count. Right: horizontal thread angle.

Synchrosqueezed (SS) transforms

wavelets		SS wavelet (SSWT)
wave packets	$+\mathrm{SS} =$	SS wave packet (SSWPT)
general curvelets		SS curvelet (SSCT)

SS for sharpened representation

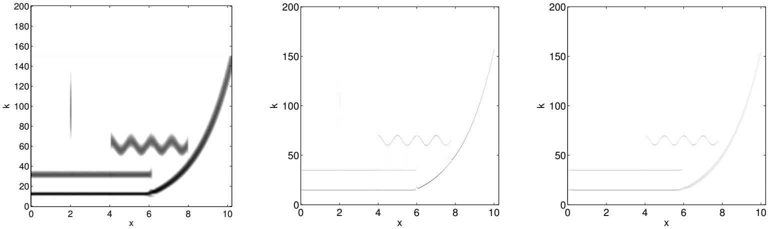


Figure : CWT, 1D SSWT and 1D SSWPT of the synthetic benchmark signal.

Theory for SS transforms

- A non-linear wave $s(x) = \alpha(x)e^{2\pi i\phi(x)}$, a wavelet or a wave packet $w_{ab}(x)$, define a transform:

$$W_s(a,b) = \langle s(x), w_{ab}(x)\rangle = \int s(x)\overline{w_{ab}(x)}\,\mathrm{d}x.$$

- Main results:
  - 1D: Daubechies et al, ACHA 11, Yang, ACHA 14

$$\omega_s(a,b) = \frac{\partial_b W_s(a,b)}{2\pi i W_s(a,b)} \approx \phi'(b)$$

  - 2D: Yang and Ying SIIMS 13 and SIMA 14

$$\omega_s(a,b) = \frac{\nabla_b W_s(a,b)}{2\pi i W_s(a,b)} \approx \nabla\phi(b)$$

Synchrosqueezing for sharpened representation:

$$\mathcal{T}_s(\omega, b) = \int_{\{a: W_s(a,b) \neq 0\}} W_s(a,b)\delta(\omega_s(a,b) - \omega)\,\mathrm{d}a.$$

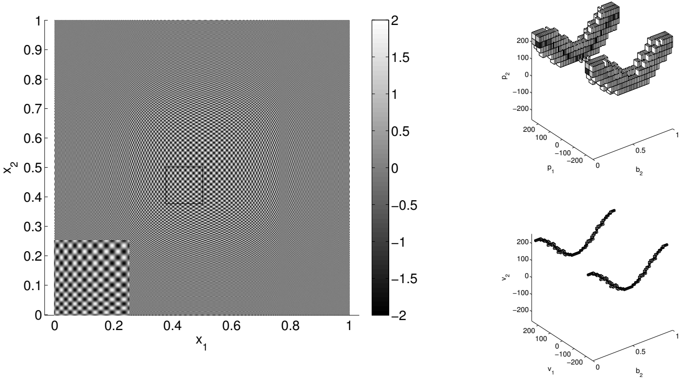


Figure : An example of a superposition of two 2D waves using 2D SSWPT.

Difference of wavelets and wave packets

Wavelet $w_{ab}(x)$: the essential support of $\widehat{w_{ab}}(\xi)$ is $\mathcal{O}(a)$.

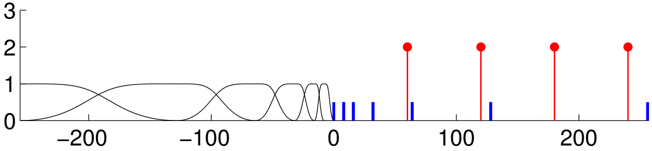


Figure : In the frequency domain: Wavelet tiling (blue); Sampling bump functions (black); Fourier transforms of plane waves (red).

Wave packets $w_{ab}(x)$: the essential support of $\widehat{w_{ab}}(\xi)$ is $\mathcal{O}(a^s)$ for a fixed $s \in [1/2, 1]$.

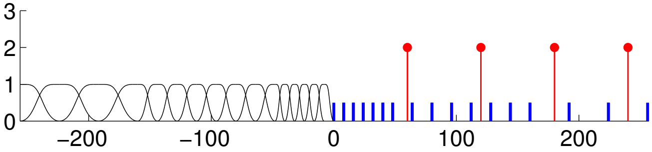


Figure : Wave packets.

Difference of SS wavelets and SS wave packets

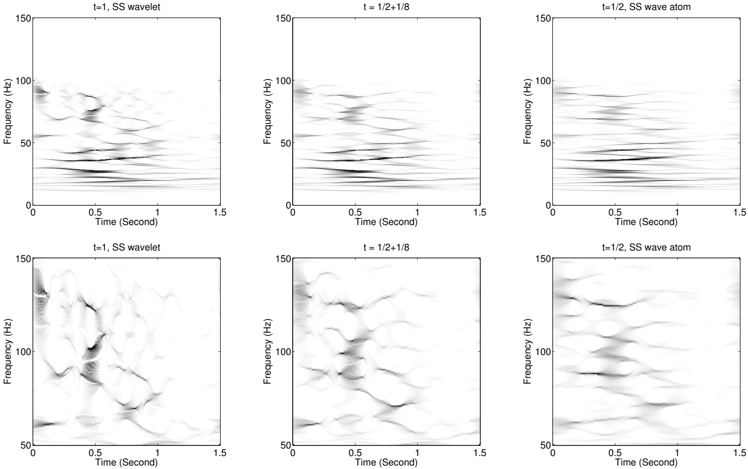


Figure : Seismic trace benchmark signal: 1D SSWT; 1D SSWPT $s = 0.625$; 1D SSWPT $s = 0.5$. Top: whole domain. Bottom: high frequency part.

Difference of 2D wave packets and 2D general curvelets

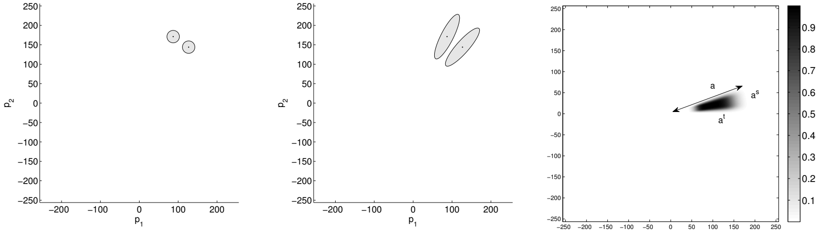


Figure : Essential support of the Fourier transform of: continuous wave packets; continuous general curvelets; a discrete general curvelet with parameters (s, t), roughly of size $a^s \times a^t$.

Difference of 2D SSWPT and 2D SSCT

Usually $s = t$ is better than $s < t$, except for the banded wave-like components.

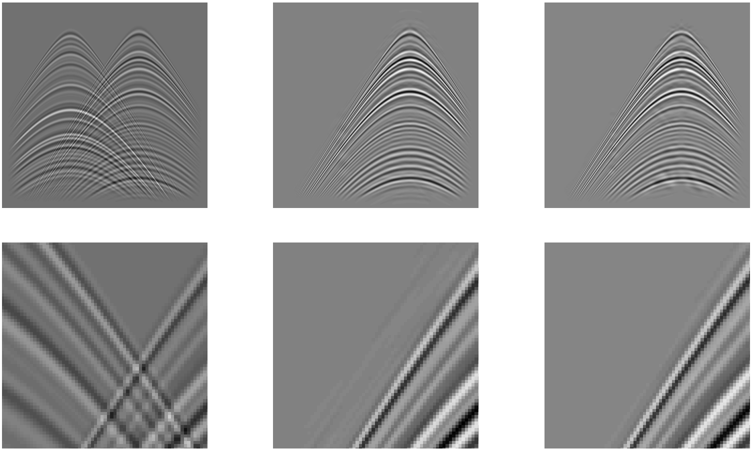


Figure : Left: A superposition of two banded waves; Middle: 2D SSWPT; Right: 2D SSCT.

Robustness properties of SSTs

1D SSWT[1][2]

- Bounded perturbation;
- Gaussian random noise (colored).

1D, 2D SSWPT, SSCT[3]

- Bounded perturbation;
- Gaussian random noise (colored);
- Possible compactly supported in space;
- Emphasize on how to realize better robustness.

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[1] G. Thakur, E. Brevdo, N.S. Fuckar, and H.-T. Wu, "The Synchrosqueezing algorithm for time-varying spectral analysis: robustness properties and new paleoclimate applications", Signal Processing, 93(5):1079–1094, 2013

[2] Y.-C. Chen, M.-Y. Cheng, H.-T. Wu, "Nonparametric and adaptive modeling of dynamic periodicity and trend with heteroscedastic and dependent errors", JRSS(B), 76(3): 651-682, 2014

[3] Yang and L. Ying, arXiv:1410.5939, 14.

Robustness properties of SSTs

Smaller scaling parameter s in the SSWPT, better robustness.

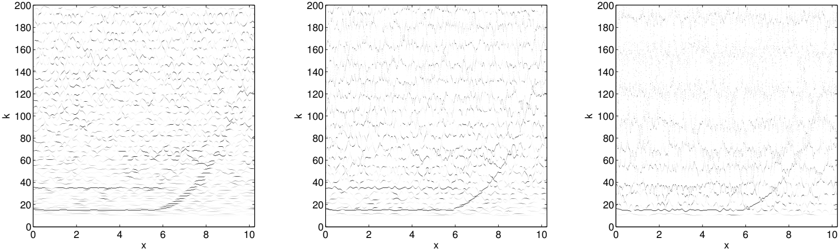

Figure : Noisy synthetic benchmark signal. From left to right: $s = 0.625$, $s = 0.75$, and $s = 0.875$.

Robustness properties of SSTs

Higher redundancy in the time-frequency transform, better robustness.

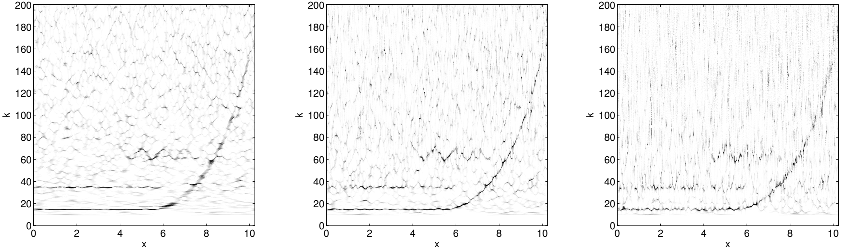

Figure : 16 times redundancy. From left to right: $s = 0.625$, $s = 0.75$, and $s = 0.875$.

Laughing voice

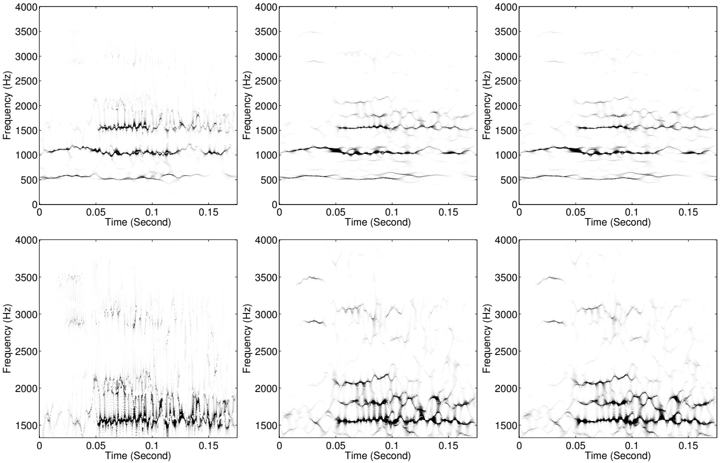

Figure : From left to right: $s = 1$ (SSWT); $s = 0.75$; $s = 0.625$. Top: whole domain. Bottom: high frequency part.

Volcanic signal tremor

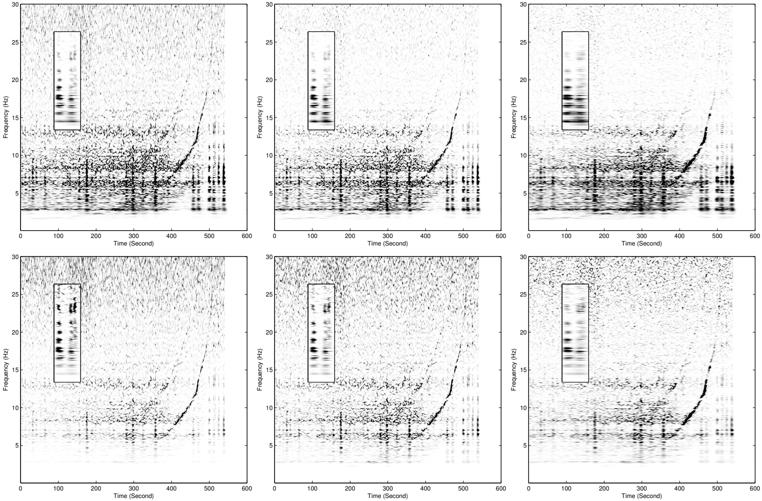

Figure : From left to right: $s = 1$ (SSWT); $s = 0.75$; $s = 0.625$. Top: Normal SST. Bottom: Enhance the energy in the high frequency part.

Microseismic signal

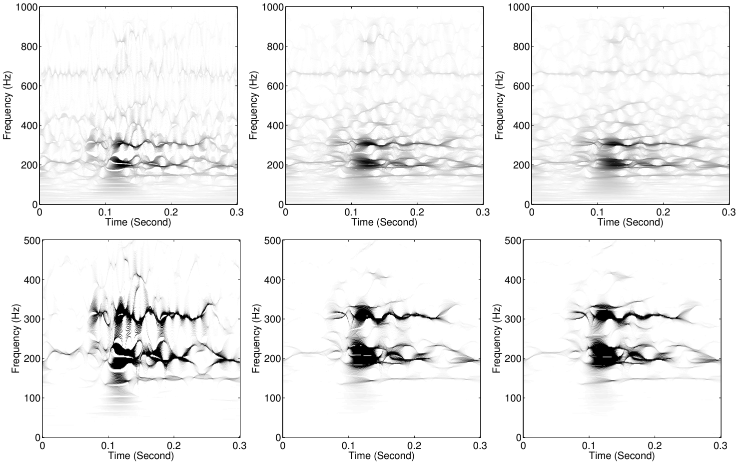

Figure : From left to right: $s = 1$ (SSWT); $s = 0.75$; $s = 0.625$. Top: whole domain. Bottom: low frequency part.

Tohoku mega-earthquake signal

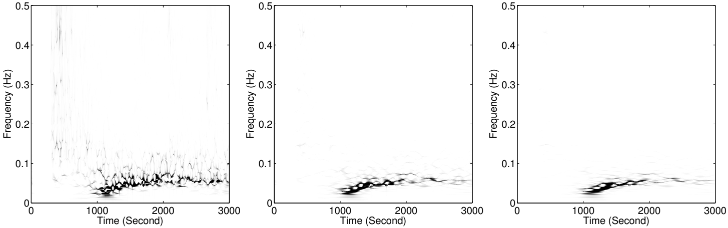

Figure : From left to right: $s = 1$ (SSWT); $s = 0.75$; $s = 0.625$.

SynLab: a MATLAB toolbox

- Available at http://web.stanford.edu/∼haizhao/Codes.htm.
- 1D SS Wave Packet Transform [4]
- 2D SS Wave Packet/Curvelet Transform [5][6]

Applications:

- Geophysics: seismic wave field separation and ground-roll removal.
- Atomic crystal image analysis.
- Art forensic.

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[4] Synchrosqueezed Wave Packet Transforms and Diffeomorphism Based Spectral Analysis for 1D General Mode Decompositions, Applied and Computational Harmonic Analysis, 2014.

[5] Synchrosqueezed Wave Packet Transform for 2D Mode Decomposition,SIAM Journal on Imaging Science, 2013.

[6] Synchrosqueezed Curvelet Transform for 2D Mode Decomposition, SIAM Journal on Mathematical Analysis, 2014.