Synchrosqueezed Transforms and Applications

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Medical study (Y., ACHA, 14)

- A superposition of two ECG signals.

$$f(t) = \alpha_1(t)s_1(2\pi\phi_1(t)) + \alpha_2(t)s_2(2\pi\phi_2(t)).$$

- Spike wave shape functions $s_1(t)$ and $s_2(t)$.

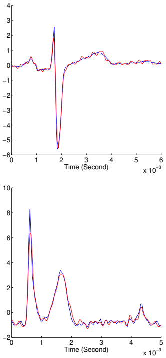


Figure : Complicated wave shape functions.

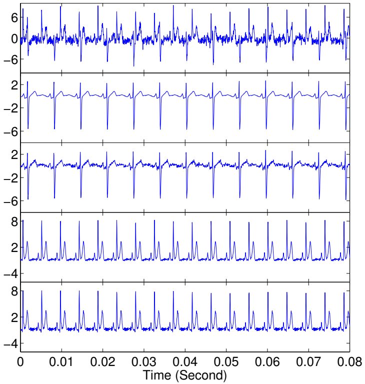


Figure : Good decomposition.

Geophysics (Y. and Ying, SIIMS 13, SIMA 14)

- A superposition of several wave fields.
- Nonlinear components, bounded supports.

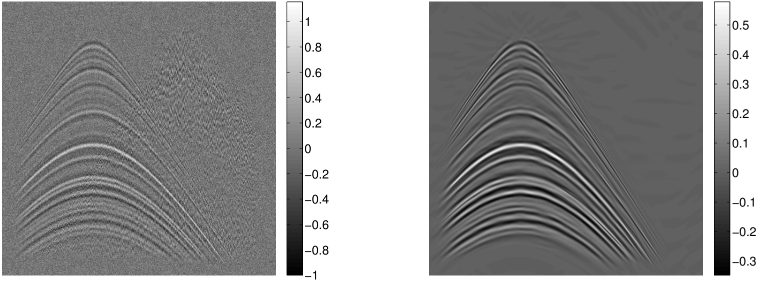
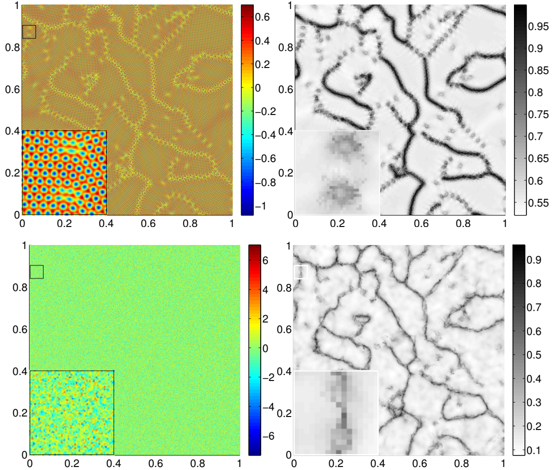


Figure : One target component with structure noise and Gaussian random noise. Courtesy of Fomel and Hu for providing data.

Materials science (Y., Lu and Ying, preprint)

Atomic crystal analysis

- Observation: an assemblage of wave-like components;
- Goal: Crystal segmentation, crystal rotations, crystal defects, crystal deformations.



Art forensics (Y., Lu, Brown, Daubechies, Ying, preprint)

Painting canvas analysis

- Observation: a superposition of wave-like components;
- Goal: count threads and estimate texture deformation.

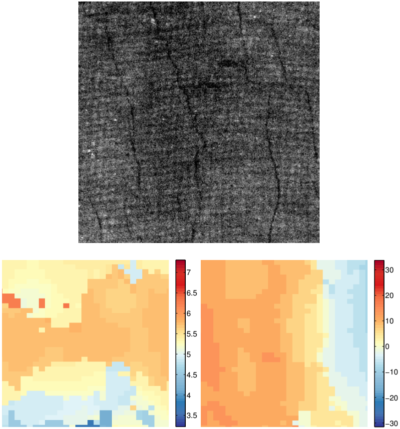


Figure : Top: a X-ray image of canvas. Left: horizontal thread count. Right: horizontal thread angle.

1D mode decomposition

Known: A superposition of wave-like components

$$f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) e^{2\pi i N_k \phi_k(t)}.$$

Unknown: Number K, components $f_k(t)$, smooth instantaneous amplitudes $\alpha_k(t)$, smooth instantaneous frequencies $N_k \phi'_k(t)$.

Existing methods:

- Empirical mode decomposition methods (Huang et al. 98, 09);
- Synchrosqueezed wavelet transform (Daubechies et al. 09, 11); Synchrosqueezed wave packet transform (Y. 14);
- Data-driven time-frequency analysis (Hou et al. 11, 12, 13);
- Regularized nonstationary autoregression (Fomel 13);

1D wave packets

Given a mother wave packet $w(t)$ and a scaling parameter $s \in (1/2, 1)$, the family of wave packets $\{w_{ab}(t) : a \geq 1, b \in \mathbb{R}\}$ is defined as

$$w_{ab}(t) = a^{s/2} w(a^s(t-b)) e^{2\pi i (t-b) a},$$

or equivalently, in the Fourier domain as

$$\widehat{w_{ab}}(\xi) = a^{-s/2} e^{-2\pi i b \xi} \widehat{w}(a^{-s}(\xi - a)).$$

1D wave packet transform

The 1D wave packet transform of a function $f(t)$ is a function

$$W_f(a,b) = \langle w_{ab}, f \rangle = \int \overline{w_{ab}(t)} f(t) dt$$

for $a \geq 1, b \in \mathbb{R}$.

A simple example

A plane wave with an instantaneous frequency N:

$$f(t) = e^{2\pi i N t}.$$

Its wave packet transform:

$$\begin{aligned} W_f(a,b) &= \int_{\mathbb{R}} e^{2\pi i N t} a^{s/2} \overline{w(a^s(t-b))} e^{-2\pi i (t-b) a} \, \mathrm{d}t \\ &= a^{-s/2} e^{2\pi i N b} \hat{w}(a^{-s}(N-a)). \end{aligned}$$

The oscillation of $W_f(a,b)$ in b reveals N:

$$\frac{\partial_b W_f(a,b)}{2\pi i W_f(a,b)} = \frac{a^{-s/2} \partial_b e^{2\pi i N b} \hat{w}(a^{-s}(N-a))}{2\pi i a^{-s/2} e^{2\pi i N b} \hat{w}(a^{-s}(N-a))} = N.$$

Definition: Instantaneous frequency estimate

$$\omega_f(a,b) = \frac{\partial_b W_f(a,b)}{2\pi i W_f(a,b)}$$

for $W_f(a,b) \neq 0$.

Definition: Synchrosqueezed wave packet transform (SSWPT)

$$\mathcal{T}_f(\omega,b) = \int_{\mathbb{R}} |W_f(a,b)|^2 \delta(\Re\omega_f(a,b) - \omega)\, \mathrm{d}a.$$

Comparison of supports

A plane wave $f(t) = e^{2\pi i N t}$, for a fixed b,

$$\operatorname{supp} W_f(a,b) \approx (N - N^s, N + N^s);$$
$$\operatorname{supp} \mathcal{T}_f(\omega,b) \text{ concentrates at } \omega = N.$$

SS for sharpened representation

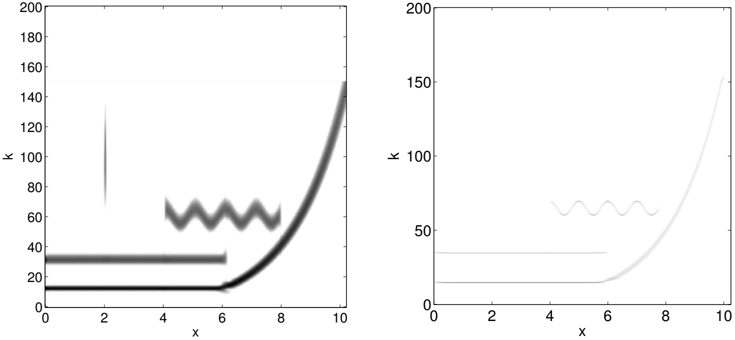


Figure : The supports of the 1D wave packet transform and 1D SSWPT of a synthetic benchmark signal.

Theory of 1D SSWPT

Theorem: (Y. 14 ACHA)

If

$$f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) e^{2\pi i N_k \phi_k(t)}$$

and $f_k(t)$ are well-separated, then

- $\mathcal{T}_f(a, b)$ has well-separated supports Z_k concentrating $(N_k \phi_k'(b), b)$;
- $f_k(t)$ can be accurately recovered by applying an inverse transform on $\mathcal{I}_{Z_k}(a, b)\mathcal{T}_f(a, b)$.

where $\mathcal{I}_{Z_k}(a, b)$ is an indication function.

Robustness properties of 1D SSWPT

- Bounded perturbation;
- Gaussian random noise (colored);
- Possible compactly supported in space.

Theorem: (Y. and Ying, 14, preprint)

- A non-linear wave $f(x) = \alpha(x)e^{2\pi i N\phi(x)}$, $\phi(x) = O(1)$;
  A zero mean Gaussian random noise e with covariance ϵ_1^q for some $q > 0$;
  A wave packet $w_{ab}(x)$ compactly supported in the Fourier domain;
- Main results: if $s(x) = f(x) + e$, then after thresholding, with a probability at least

$$\left(1 - e^{-O(N^{2-3s}\epsilon_1^{-q})}\right)\left(1 - e^{-O(N^{-2-s}\epsilon_1^{-q})}\right),$$

$$\omega_s(a,b) = \frac{\partial_b W_s(a,b)}{2\pi i W_s(a,b)} \approx N\phi'(b)$$

Properties of 1D SSWPT[1][2]

- When $s = 1$, wave packets become wavelets;
- When $s = 1/2$, wave packets become wave atoms;
- Larger s, more accurate to estimate instantaneous frequencies;
- Smaller s, more robust to estimate instananeous frequencies;
- Smaller s, better resolution to distinguish wave-like components in the high frequency domain.
- Smaller s, better for the general mode decomposition problem:

$$
\begin{aligned}
f(t) = \sum_{k=1}^{K} f_k(t) &= \sum_{k=1}^{K} \alpha_k(t) s_k(2\pi N_k \phi_k(t)) \\
&= \sum_{k=1}^{K} \alpha_k(t) \sum_n \widehat{s_k}(n) e^{2\pi i N_k n \phi_k(t)}
\end{aligned}
$$

[1] Y. ACHA, 14.

[2] Y. and Ying, arXiv:1410.5939, 14.

Difference of wavelets and wave packets

The size of the essential support of $\widehat{w_{ab}}(\xi)$ is $\mathcal{O}(a^s)$.

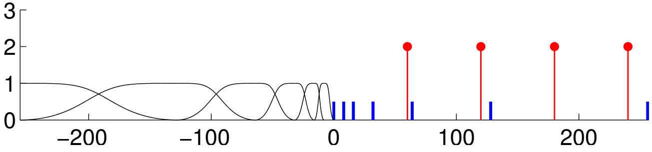


Figure : In the frequency domain: $s = 1$, wavelet tiling (blue); Sampling bump functions (black); Fourier transforms of plane waves (red).

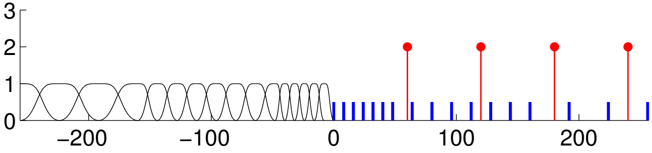


Figure : $s < 1$, wave packets.

Difference of SS wavelets and SS wave packets

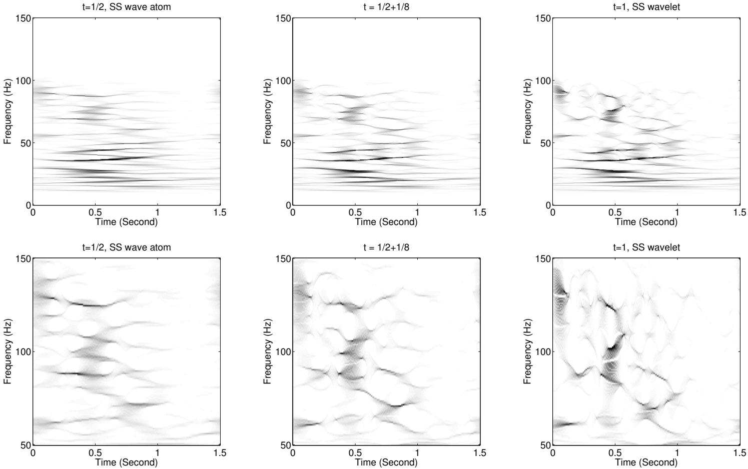


Figure : Seismic trace benchmark signal: $s = 0.5$; $s = 0.625$; $s = 1$. Top: whole domain. Bottom: high frequency part.

Robustness properties of 1D SSTs

Smaller scaling parameter s in the SSWPT, better robustness.

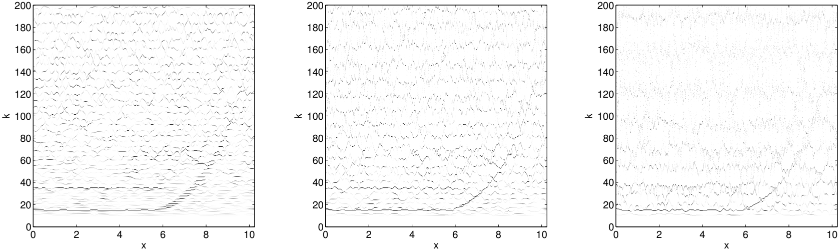


Figure : Noisy synthetic benchmark signal. From left to right: $s = 0.625$, $s = 0.75$, and $s = 0.875$.

Robustness properties of 1D SSTs

Higher redundancy in the time-frequency transform, better robustness.

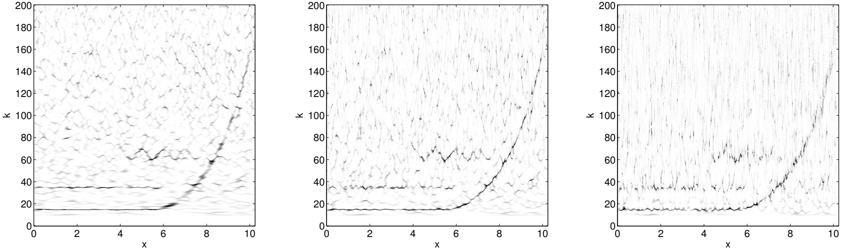


Figure : 16 times redundancy. From left to right: $s = 0.625$, $s = 0.75$, and $s = 0.875$.

Volcanic signal tremor

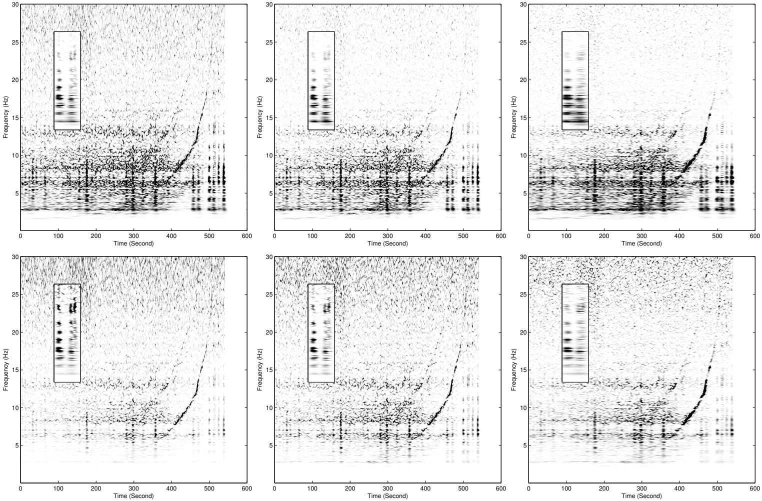


Figure : From left to right: $s = 1$ (SSWT); $s = 0.75$; $s = 0.625$. Top: Normal SST. Bottom: Enhance the energy in the high frequency part.

2D Synchrosqueezed (SS) transforms

2D wave packets	$+\mathrm{SS}=$	2D SS wave packet (SSWPT)
2D general curvelets		2D SS curvelet (SSCT)

2D wave packets

2D wave packets $\{w_{ab}(x) : a, b \in \mathbb{R}^2, |a| \geq 1\}$ are defined as

$$w_{ab}(x) = |a|^s w(|a|^s(x - b))e^{2\pi i(x-b)\cdot a},$$

or equivalently in Fourier domain

$$\widehat{w_{ab}}(\xi) = |a|^{-s} e^{-2\pi i b\cdot\xi}\hat{w}(|a|^{-s}(\xi - a)).$$

Notations:

1. The scaling matrix
$$A_a = \begin{pmatrix} a^t & 0 \\ 0 & a^s \end{pmatrix}.$$
2. The rotation angle θ and rotation matrix
$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
3. A unit vector $e_\theta = (\cos\theta, \sin\theta)^T$ with a rotation angle θ.

2D general curvelets

2D general curvelets $\{w_{a\theta b}(x), a \in [1, \infty), \theta \in [0, 2\pi), b \in \mathbb{R}^2\}$ are defined as

$$w_{a\theta b}(x) = a^{\frac{t+s}{2}} e^{2\pi i a(x-b)\cdot e_\theta} w(A_a R_\theta^{-1}(x-b)),$$

or equivalently in Fourier domain

$$\widehat{w_{a\theta b}}(\xi) = \widehat{w}(A_a^{-1} R_\theta^{-1}(\xi - a \cdot e_\theta)) e^{-2\pi i b\cdot\xi} a^{-\frac{t+s}{2}}.$$

2D wave packets and 2D general curvelets

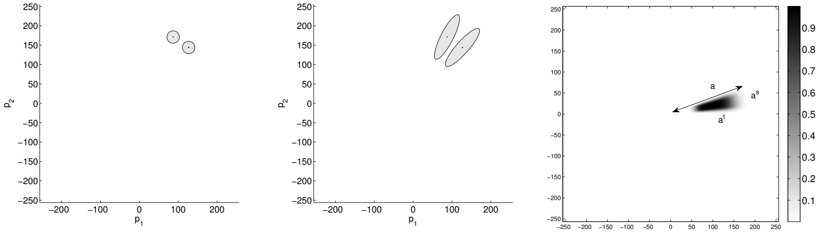


Figure : Essential support of the Fourier transform of: continuous wave packets; continuous general curvelets; a discrete general curvelet with parameters (s, t), roughly of size $a^s \times a^t$.

Theory for 2D SS wave packet transforms

Theroem 1: (Y. and Ying SIIMS 13)

A non-linear wave $f(x) = \alpha(x)e^{2\pi i\phi(x)}$, a wave packet $w_{ab}(x)$, define a transform:

$$W_f(a,b) = \langle s(x), w_{ab}(x)\rangle = \int s(x)\overline{w_{ab}(x)}\,\mathrm{d}x.$$

$$\omega_f(a,b) = \frac{\nabla_b W_f(a,b)}{2\pi i W_f(a,b)} \approx \nabla\phi(b)$$

Theorem 2: (Y. and Ying preprint 14)

A zero mean Gaussian random noise e with covariance ϵ_1^q for some $q > 0$. If $s(x) = f(x) + e$, then after thresholding, with a probability at least

$$\left(1 - e^{-O(N^{2-2s}\epsilon_1^{-q})}\right)\left(1 - e^{-O(N^{-2s}\epsilon_1^{-q})}\right)\left(1 - e^{-O(N^{-2}\epsilon_1^{-q})}\right),$$

$$\omega_s(a,b) = \frac{\nabla_b W_s(a,b)}{2\pi i W_s(a,b)} \approx N\nabla\phi(b)$$

Theory for 2D SS curvelet transforms

Theroem 1: (Y. and Ying SIMA 14)

A non-linear wave $f(x) = \alpha(x)e^{2\pi i\phi(x)}$, a general curvelet $w_{a\theta b}(x)$, define a transform:

$$W_f(a,\theta,b) = \langle s(x), w_{a\theta b}(x)\rangle = \int s(x)\overline{w_{a\theta b}(x)}\,\mathrm{d}x.$$

$$\omega_f(a,\theta,b) = \frac{\nabla_b W_f(a,\theta,b)}{2\pi i W_f(a,\theta,b)} \approx \nabla\phi(b)$$

Theroem 2: (Y. and Ying preprint 14)

A zero mean Gaussian random noise e with covariance ϵ_1^q for some $q > 0$. If $s(x) = f(x) + e$, then after thresholding, with a probability at least

$$\left(1 - e^{-O(N^{2-2t}\epsilon_1^{-q})}\right)\left(1 - e^{-O(N^{-2s}\epsilon_1^{-q})}\right)\left(1 - e^{-O(N^{-2}\epsilon_1^{-q})}\right),$$

$$\omega_f(a,\theta,b) = \frac{\nabla_b W_f(a,\theta,b)}{2\pi i W_f(a,\theta,b)} \approx \nabla\phi(b)$$

Synchrosqueezing for sharpened representation:

$$\mathcal{T}_f(\omega, b) = \int_{\{a: W_f(a,b) \neq 0\}} W_f(a, b)\delta(\omega_f(a, b) - \omega)\,\mathrm{d}a.$$

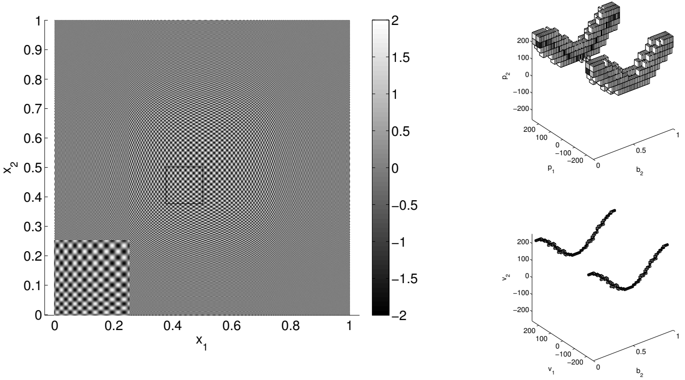


Figure : An example of a superposition of two 2D waves using 2D SSWPT.

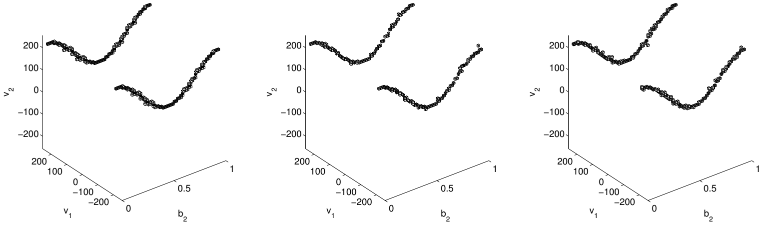


Figure : $\mathcal{T}_f(\omega, b)$ of the same example in the last figure. Left: noiseless. Middle: SNR$= 3$. Right: SNR$= -3$.

Difference of 2D SSWPT and 2D SSCT

Usually $s = t$ is better than $s < t$, except for the banded wave-like components.

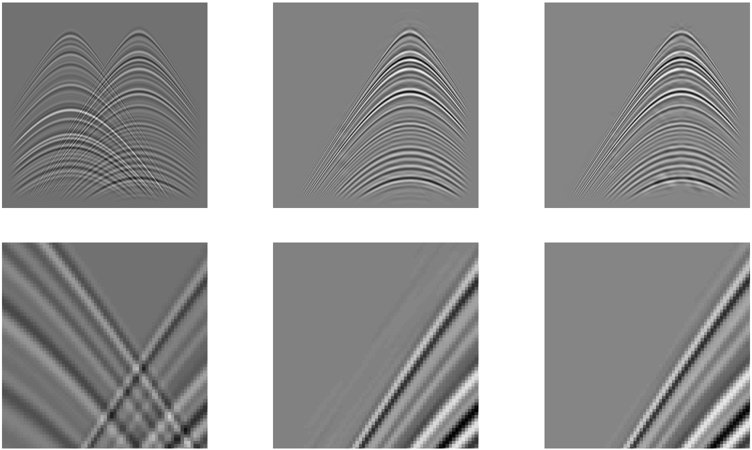


Figure : Left: A superposition of two banded waves; Middle: 2D SSWPT; Right: 2D SSCT.

SynLab: a MATLAB toolbox

- Available at http://web.stanford.edu/∼haizhao/Codes.htm.
- 1D SS Wave Packet Transform [3]
- 2D SS Wave Packet/Curvelet Transform [4][5]

Applications:

- Geophysics: seismic wave field separation and ground-roll removal.
- Atomic crystal image analysis.
- Art forensic.

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[3] Synchrosqueezed Wave Packet Transforms and Diffeomorphism Based Spectral Analysis for 1D General Mode Decompositions, Applied and Computational Harmonic Analysis, 2014.

[4] Synchrosqueezed Wave Packet Transform for 2D Mode Decomposition,SIAM Journal on Imaging Science, 2013.

[5] Synchrosqueezed Curvelet Transform for 2D Mode Decomposition, SIAM Journal on Mathematical Analysis, 2014.